$\lambda(t, x, \xi)$

Breaking Homogeneity: From Lévy to Feller–Sato Processes

Pseudodifferential Operators and Markovian Potential Theory

T. Zamrik

ODIN PRESS

Breaking Homogeneity: From Lévy to Feller–Sato Processes

Published by Odin Press.
`https://odinpress.org`
This book was typeset in LaTeX in Latin Modern Roman by Odin Press from a manuscript submitted by the author.
Paperback version, trim size 6 × 9 inches.
ISBN: forthcoming (Bowker pending)
Printed and bound in the United States of America.

9 8 7 6 5 4 3 2 1 0

Preface

In 1966, Mark Kac asked whether one can hear the shape of a drum. The question is this: if you listen only to the frequencies at which a drumhead vibrates, can you determine its geometry? The answer, worked out over the following decades, is: almost, but not quite. The spectrum of the Laplacian encodes a great deal about a domain, but not everything.

This book asks the probabilist's version of the same question. Given a Markov process, what does its generator tell you about its trajectories? The generator is a pseudodifferential operator, and its symbol — the function $\lambda(t, x, \xi)$ in the Fourier variable ξ, possibly depending on time t and position x — encodes the infinitesimal structure of the process in the same way that eigenvalues encode the shape of a drum. The symbol is the object this book is about.

The symbol takes four forms, each corresponding to a distinct class of processes. In its simplest form, $\lambda(\xi)$, it is constant in time and space: this is the characteristic exponent of a Lévy process, and the Lévy–Khintchine theorem gives the complete classification. When time-dependence is introduced, the symbol becomes $\lambda(t, \xi)$, and the associated processes are the additive and self-similar processes studied by Sato. When spatial dependence replaces temporal dependence, the symbol becomes $\lambda(x, \xi)$: this is the domain of Feller processes, characterised by Courège's theorem as precisely those Markov processes whose generators satisfy the positive maximum principle. The full symbol $\lambda(t, x, \xi)$, depending on both time and position, governs the Feller–Sato class — the most general family of jump-driven Markov processes with a rich potential theory.

The four parts of this book develop these four cases in order, each building on the last. Part I treats Lévy processes: Chapter 1 introduces infinite divisibility and the Lévy–Khintchine representation; Chapter 2 establishes the generator as a pseudodifferential operator and develops the theory of Bernstein functions and Bochner subordination; Chapter 3 develops Markovian potential theory — resolvents, excessive functions, capacity, and Hunt's theorem; Chapter 4 is devoted to the Wiener–Hopf factorisation and the ladder subordinators; Chapter 5 treats scale functions for spectrally negative processes. Part II breaks temporal homogeneity: Chapters 6–8 develop additive processes, self-similar processes, and potential theory for Sato processes, showing which results from Part I survive the loss of stationarity and which require new arguments. Part III breaks spatial homogeneity: Chapters 9–12 treat Feller semigroups, Courège's theorem, existence results via the martingale problem and Levi parametrix, and potential theory with variable symbol. Part IV combines both inhomogeneities: Chapters 13–15 develop the full symbol calculus, evolution systems of operators, and the hero result of the book — heat kernel estimates for the Feller–Sato class in the spirit of Knopova, Schilling, and Wang. Chapter 16 closes with potential theory at full generality: the Dirichlet problem for non-local operators, exit distributions, polar sets, and the Wiener criterion.

Three appendices support the main text. Appendix A collects the functional analysis and semigroup theory used throughout. Appendix B develops the symbol calculus of pseudodifferential operators systematically. Appendix C discusses numerical methods for the symbol: quadrature for the Lévy–Khintchine integral, FFT-based discretisation of $\mathrm{Op}[\lambda(x, \xi)]$, and symbol-based heat kernel bounds. The book is selfcontained via these appendices; the prerequisites are undergraduate probability, functional analysis, and real analysis.

The single structural feature that ties all four parts together is the following: for every fixed value of the parameters (t, x), the map $\xi \mapsto \lambda(t, x, \xi)$ is a continuous negative definite function. This property, which originates in the Lévy–Khintchine theorem of Chapter 1, is preserved under all the deformations introduced in Parts II, III, and IV. It is the constraint that makes the symbol the right object to study, and the reason the answer to the probabilist's version of Kac's question is richer and more complete than the analyst's.

T. Zamrik
April 2025

Contents

Chapter 1

Infinite Divisibility and the Lévy–Khintchine Representation

1.1 Introduction

The classical central limit theorem identifies a privileged class of distributions — the Gaussian — as the universal attractor under summation and scaling. Behind this universality lies a structural property: the Gaussian distribution can be written as a product of identical factors in the space of characteristic functions for every positive integer n. This chapter develops the precise formulation of that property, extends it to the full class of distributions sharing it, and extracts from them the central analytic object of this book.

The theory of infinite divisibility was developed by Bruno de Finetti in the 1930s and brought to its definitive form by Paul Lévy and Andrei Khintchine. The representation theorem that bears their names, and the pathwise decomposition of the associated processes into independent components due to Lévy and Itô, constitute the foundation on which everything that follows is built. The standard references are Sato [48], Bertoin [6], Jacob [27], and Applebaum [1]; we follow their conventions throughout, with the emphasis on the characteristic exponent as the primary object.

A single formula emerges at the end of this chapter and will not change form until Chapter 12. It is

$$\lambda(\xi) = ib\xi - \tfrac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{i\xi y} - 1 - i\xi y\,\mathbf{1}_{|y|\le 1}\right)\nu(dy), \tag{1.1}$$

the characteristic exponent of an infinitely divisible distribution. In Parts II, III, and IV this formula will be progressively deformed: the triplet (b, σ^2, ν) will acquire dependence on time, on space, and finally on both simultaneously. But the structure of (1.1) persists throughout as the skeleton of the symbol of a pseudodifferential operator.

1.2 Infinite Divisibility

Definition 1.1 (Infinitely divisible distribution)**.** A probability measure μ on $\mathbb{R}$ is *infinitely divisible* if for every $n \in \mathbb{N}$ there exists a probability measure μ_n on $\mathbb{R}$ such that

$$\mu = \underbrace{\mu_n * \mu_n * \cdots * \mu_n}_{n \text{ factors}}, \tag{1.2}$$

where $*$ denotes convolution.

In terms of characteristic functions, (1.2) is the condition that

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}} e^{i\xi x}\,\mu(dx) = \left(\widehat{\mu_n}(\xi)\right)^n \tag{1.3}$$

for some characteristic function $\widehat{\mu_n}$, for every $n \ge 1$. The Gaussian measure $\mathcal{N}(0,\sigma^2)$ is infinitely divisible: $\widehat{\mu}(\xi) = e^{-\sigma^2\xi^2/2}$ and $(e^{-\sigma^2\xi^2/2})^{1/n} = e^{-\sigma^2\xi^2/(2n)}$ is the characteristic function of $\mathcal{N}(0,\sigma^2/n)$. The Poisson measure with parameter $\lambda_0 > 0$ and the Cauchy distribution are infinitely divisible. The uniform distribution on $[0, 1]$ is not.

A fundamental structural consequence is that the characteristic function of an infinitely divisible measure never vanishes, which allows the characteristic exponent to be defined globally.

Proposition 1.2 (Non-vanishing of $\widehat{\mu}$)**.** *If μ is infinitely divisible then $\widehat{\mu}(\xi) \neq 0$ for all $\xi \in \mathbb{R}$, and the function $\lambda(\xi) = \log \widehat{\mu}(\xi)$ (principal branch) is a well-defined continuous function.*

Proof. Suppose $\widehat{\mu}(\xi_0) = 0$ for some ξ_0. For each n, we have $\widehat{\mu}(\xi_0) = \widehat{\mu_n}(\xi_0)^n = 0$, so $\widehat{\mu_n}(\xi_0) = 0$. Now use the standard inequality: for any probability measure ν,

$$|\widehat{\nu}(\xi)|^2 \leq \tfrac{1}{2}\big(1 + \mathrm{Re}\,\widehat{\nu * \tilde{\nu}}(\xi)\big) \tag{1.4}$$

where $\tilde{\nu}$ is the reflection of ν. Applying this to μ_n and tracking the limit $n \to \infty$ shows that $\widehat{\mu}(\xi_0) = 0$ forces all nearby characteristic functions to vanish as well, contradicting continuity of $\widehat{\mu}$ at $\xi_0 = 0$ where $\widehat{\mu}(0) = 1$. The full argument is given in Sato [48] Lemma 7.5. ☐

Remark 1.3. Proposition 1.1 ensures that $\lambda(\xi) = \log \widehat{\mu}(\xi)$ is well-defined and continuous. The Lévy–Khintchine theorem characterises precisely which continuous functions arise this way.

1.3 The Lévy Measure and the Lévy Triplet

Before stating the main theorem, we identify the analytic ingredients that parametrise all infinitely divisible distributions.

Definition 1.4 (Lévy measure)**.** A σ-finite measure ν on $\mathbb{R} \setminus \{0\}$ is a *Lévy measure* if

$$\int_{\mathbb{R}\setminus\{0\}} \min(1, y^2)\, \nu(dy) < \infty. \tag{1.5}$$

Chapter 3

Markovian Potential Theory: Resolvents, Excessive Functions, Capacity

3.1 Introduction

Classical potential theory begins with harmonic and superharmonic functions, objects defined analytically through the Laplacian. The probabilistic revolution of the mid-twentieth century, carried through by Hunt, Doob, Blumenthal, and Getoor, revealed that these analytic notions have intrinsic probabilistic counterparts: harmonic functions correspond to martingales, superharmonic functions to supermartingales, and the Laplacian itself to the generator of Brownian motion. The monograph of Blumenthal and Getoor [7] systematised this dictionary and extended it to arbitrary Markov processes. For Lévy processes the picture is especially clean: the homogeneous symbol $\lambda(\xi)$ determines the spectral theory of the generator, and consequently every potential-theoretic object of the process can be computed from $\lambda(\xi)$ via Fourier analysis.

This chapter builds the potential-theoretic apparatus that will be deformed in Parts II–IV of the book when $\lambda(\xi)$ is replaced by a symbol $\lambda(t, x, \xi)$ depending on position and time. We define the resolvent family $\{\mathscr{R}_q\}_{q>0}$, establish the resolvent identity, and show that $\mathscr{R}_q$ inverts the shifted generator $q - \mathscr{L}$. We then introduce q-excessive functions as the probabilistic counterpart of q-superharmonic functions, and prove the Riesz decomposition that factorises every excessive function into a potential part and a harmonic part. The potential kernel U and its density (the Green's function) are constructed via Fourier analysis on $\mathbb{R}^d$. We define capacity through equilibrium measures, establish Hunt's theorem connecting capacity to polarity, prove that capacity is a Choquet capacity, and characterise polar sets and semipolar sets probabilistically.

Blumenthal and Getoor [7], Port and Stone [45], and Chung and Walsh [13] are the primary references for this chapter. Jacob's second volume [28] provides the Fourier-analytic perspective linking the symbol $\lambda(\xi)$ to potential-theoretic quantities. We consistently work with a Lévy process $(X_t)_{t\geq 0}$ on $\mathbb{R}^d$ with characteristic exponent $\lambda(\xi)$ as defined in Chapter 1.

3.2 The Resolvent Family

Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$ with semigroup $\mathcal{T}_t f(x) = \mathbb{E}_x[f(X_t)]$ acting on $\mathcal{B}_b(\mathbb{R}^d)$, the bounded Borel functions. The resolvent family encodes all time-integrated information about the process.

Definition 3.1 (Resolvent Operator)**.** For $q > 0$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$, the *$q$-resolvent* $\mathscr{R}_q$ is the operator

$$\mathscr{R}_q f(x) = \mathbb{E}_x\left[\int_0^\infty e^{-qt} f(X_t)\, dt\right] = \int_0^\infty e^{-qt}\, \mathcal{T}_t f(x)\, dt. \tag{3.1}$$

The resolvent $\mathscr{R}_q f(x)$ represents the expected discounted occupation time of the process starting at x, weighted by the payoff function f, at discount rate q.

Proposition 3.2 (Boundedness and Positivity of the Resolvent)**.** *For every $q > 0$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$:*

(i) $\|\mathscr{R}_q f\|_\infty \leq q^{-1}\|f\|_\infty$.

(ii) If $f \geq 0$ *then* $\mathscr{R}_q f \geq 0$.

(iii) The map $q \mapsto q\mathscr{R}_q f(x)$ *is non-increasing for* $f \geq 0$, *with* $\lim_{q\to\infty} q\mathscr{R}_q f(x) = f(x)$ *and* $\lim_{q\to 0^+} q\mathscr{R}_q f(x) = 0$ *when the process is transient.*

Proof. Step 1 (Boundedness). By definition and $\|\mathcal{T}_t f\|_\infty \leq \|f\|_\infty$,

$$\|\mathscr{R}_q f\|_\infty \leq \int_0^\infty e^{-qt}\|\mathcal{T}_t f\|_\infty\, dt \leq \|f\|_\infty \int_0^\infty e^{-qt}\, dt = \frac{\|f\|_\infty}{q}. \tag{3.2}$$

Step 2 (Positivity). If $f \geq 0$ then $\mathcal{T}_t f(x) = \mathbb{E}_x[f(X_t)] \geq 0$ for all t and x, so the integral in (A.7) is non-negative.

Step 3 (Monotonicity in q). For $f \geq 0$, $q\mathscr{R}_q f(x) = q\int_0^\infty e^{-qt}\mathcal{T}_t f(x)\, dt$. Differentiating in q: $\partial_q[q\mathscr{R}_q f(x)] = \int_0^\infty e^{-qt}(1-qt)\mathcal{T}_t f(x)\, dt$, which is not sign-definite. In fact $q \mapsto q\mathscr{R}_q f(x)$ is non-increasing: for $p < q$, $p\mathscr{R}_p f(x) \geq q\mathscr{R}_q f(x)$ follows from the resolvent identity $\mathscr{R}_p - \mathscr{R}_q = (q-p)\mathscr{R}_p\mathscr{R}_q$ and $\mathscr{R}_p, \mathscr{R}_q \geq 0$. □

Remark 3.3. The limit $\lim_{q\to\infty} q\mathscr{R}_q f(x) = f(x)$ holds pointwise for every $f \in C_b(\mathbb{R}^d)$ and every $x \in \mathbb{R}^d$. This is the probabilistic analogue of the approximation-of-identity property of the resolvent in functional analysis; it follows from $\mathcal{T}_0 = \mathrm{Id}$ and the right-continuity of $t \mapsto \mathcal{T}_t f(x)$.

$$\lambda(\xi) \to \lambda(t, \xi) \to \lambda(x, \xi) \to \lambda(t, x, \xi)$$

T. Zamrik

Breaking Homogeneity: From Lévy to Feller–Sato Processes

In 1966, Mark Kac asked whether one can hear the shape of a drum. This book asks the probabilist's version: what does the generator of a Markov process tell you about the process itself? The answer is encoded in a single function — the symbol — and this book is its complete theory. Breaking Homogeneity traces four stages of increasing generality. Each part breaks one more symmetry: temporal homogeneity falls first (Sato processes), then spatial homogeneity (Feller processes), until the fully inhomogeneous Feller–Sato class is reached. At every stage, the symbol remains a negative definite function in the Fourier variable — the single invariant tying the entire theory together. The book covers infinite divisibility, Wiener–Hopf factorisation, Courège's theorem, the parametrix construction, and heat kernel estimates. Potential theory is developed and compared systematically across all four classes. Dr. T. Zamrik is an applied mathematician working at the intersection of stochastic analysis, probability theory, stochastic control, and partial differential equations.

Cover art: symbol progression
$\lambda(\xi) \to \lambda(t,x,\xi)$, Feller–Sato class