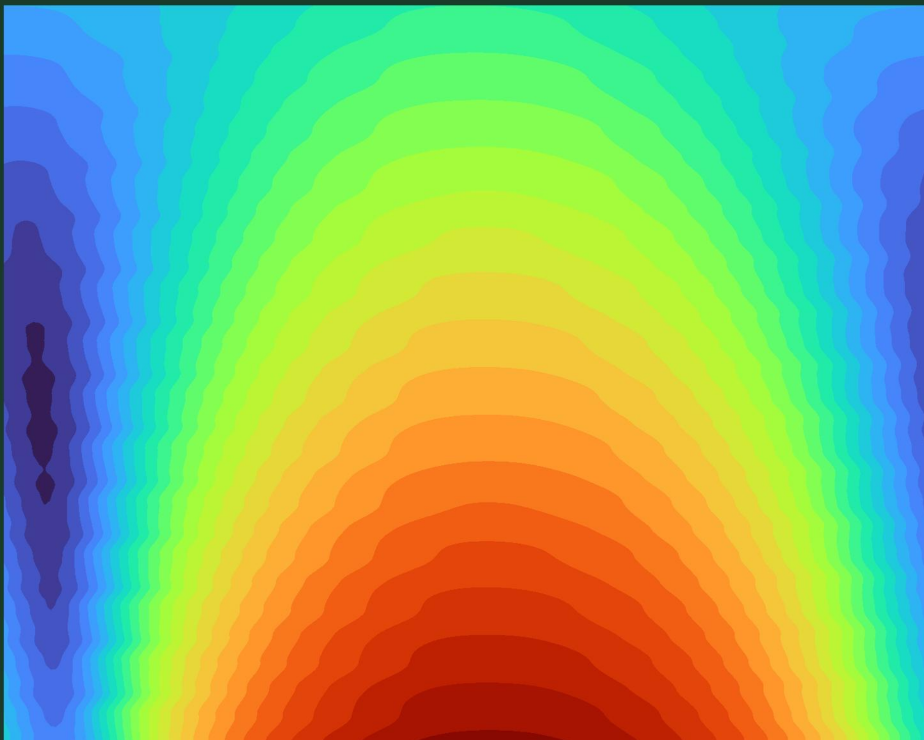

The value of re-entraining a circadian clock, computed by the resolvent method where no semigroup exists.

The Resolvent Method for PDEs

Solving Evolutionary Equations Where the Semigroup Cannot

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Preface

The standard course in evolution equations puts the semigroup at the centre — the exponential of the generator, the operator that carries an initial state forward in time — and treats the resolvent, the inverse of the shifted generator at a single complex frequency, as the scaffolding one uses to build it. I hold the opposite view, and this book is the argument for it. The resolvent is the Laplace transform of the semigroup: prior to it, in the exact sense that the semigroup is recovered from the resolvent by an inverse transform — a contour quadrature carried to machine precision. Where both objects exist, this is a matter of emphasis. I insist on it because the two do not always both exist, and it is the resolvent, not the semigroup, that survives.

That the semigroup can fail is no pathology. There is a ladder of problems, each weakening one structural assumption, on which the semigroup degrades or ceases to exist while the resolvent still carries the solution. Drop self-adjointness and the eigenfunction expansion need not converge — I once watched a natural spectral series fail to represent a solution whose evolution the resolvent contour then recovered exactly. Drop autonomy and there is no exponential operator to name; the correct object is an evolution family, built from the frozen-time resolvents. Let memory enter and one has a resolvent family with no law of exponents behind it. Drop linearity and the resolvent becomes nonlinear, and the Crandall–Liggett formula generates a nonlinear semigroup from it with no smoothness assumed. The literature treats these cases well, each in its own monograph; what it does not do is carry the single thesis — the resolvent is the primary object — across all of them, to where two assumptions fail at once.

The thesis holds by exhaustion of the ladder, the object at every rung the same one wearing less structure. Part I establishes the Laplace–Bromwich duality and evaluates the analytic semigroup of a sectorial operator by contour quadrature, then builds the Dunford–Riesz functional calculus and the maximal-regularity estimate — both statements about the resolvent alone. Part II removes self-adjointness: where the eigenvalues mislead, the pseudospectrum is the honest diagnostic, and the resolvent contour recovers what the eigenvectors cannot. Part III removes density, autonomy, and the Markov property: the resolvent survives a non-dense domain, the Acquistapace–Terreni construction assembles the propagator from frozen resolvents, and the Mittag-Leffler resolvent family solves the fractional-in-time equation where no semigroup exists at all. Part IV removes linearity: the nonlinear resolvent generates the nonlinear semigroup and solves the Hamilton–Jacobi–Bellman equation as a tower of linear resolvents. The final chapter removes two assumptions at once — nonlinear and non-autonomous — and finds the value function of a real time-inhomogeneous control problem, the timed-light correction of circadian misalignment, as a convergent product of frozen-time nonlinear resolvents, computed on measured data rather than asserted.

I have tried to be as exact about where the method stops as about where it works. The resolvent method is a parabolic phenomenon: it rests on the resolvent decaying in a sector, so that the Bromwich contour can be bent into the left half-plane and the quadrature made to converge. For a genuinely hyperbolic problem — a skew-adjoint generator, a wave equation with time-varying speed — the resolvent decays in no sector, the contour will not bend, and no product of resolvents converges. I reached for the contour out of habit on such a problem once and watched the quadrature refuse; that boundary is not a gap in the method but a theorem about it, and I name it wherever it presses against the construction.

Every numerical result was produced by the code printed in Appendix A and is reproducible from it; where a computed quantity contradicted the claim I had intended to make, I changed the claim. I report each estimate with its constant and each method with its boundary, and I have given nothing in these pages that I have not carried to its last line.

— T. Zamrik, 2026

Contents

Chapter 1

The Resolvent and the Semigroup: A Laplace Duality

Two objects compete for primacy in the theory of linear evolution equations. The first is the semigroup $e^{t\mathscr{A}}$, the solution operator that advances an initial state forward in time. The second is the resolvent $(\lambda - \mathscr{A})^{-1}$, the inverse of a shifted generator at a single complex frequency. Most expositions treat the semigroup as fundamental and the resolvent as an auxiliary computed from it. This book takes the opposite view, and this chapter states the reason. For spectral parameters to the right of the growth bound the resolvent is exactly the Laplace transform of the semigroup, and where the generator is sectorial the semigroup is recovered from the resolvent by a contour integral that a handful of resolvent solves evaluate to near-machine accuracy. The correspondence runs both ways, but it is not symmetric in what it demands: the semigroup requires an autonomous, well-posed evolution before it can be written down, while the resolvent requires only that $\lambda - \mathscr{A}$ be invertible. On the ladder of harder problems the later chapters climb — non-self-adjoint, non-autonomous, fractional-in-time, fully nonlinear — the semigroup degrades or ceases to exist while the resolvent survives.

The chapter delivers the two-way Laplace–Bromwich correspondence and a four-regime map that fixes, from the outset, where exact-in-time propagation holds and where the method's advantage ends. The first section introduces the semigroup and its generator; the second identifies the resolvent as the forward Laplace transform; the third closes the forward direction with the Hille–Yosida characterization; the fourth inverts the transform through the Bromwich integral for sectorial generators; the fifth organizes the landscape into four regimes and marks the hyperbolic boundary plainly; and the last states the thesis the rest of the book generalizes — that zero error in time is realized by the resolvent, not by a marching scheme.

1.1 Strongly Continuous Semigroups and Their Generators

Let X be a Banach space and consider the autonomous Cauchy problem $\frac{du}{dt} = \mathscr{A}u$, $u(0) = x$. Its solution operator, when it exists, is a family of bounded operators indexed by time that composes additively; this is the algebraic content of a semigroup, and strong continuity is the minimal regularity that makes the family a genuine evolution [1].

Definition 1.1 (Strongly continuous semigroup)**.** A *strongly continuous* (or C_0) semigroup on a Banach space X is a family $\{\mathscr{T}(t)\}_{t\geq 0}$ of bounded linear operators on X satisfying $\mathscr{T}(0) = I$, the functional equation $\mathscr{T}(t+s) = \mathscr{T}(t)\mathscr{T}(s)$ for all $s, t \geq 0$, and the orbit continuity $\lim_{t\downarrow 0} \mathscr{T}(t)x = x$ for every $x \in X$.

The functional equation

$$\mathscr{T}(t+s) = \mathscr{T}(t)\,\mathscr{T}(s), \qquad s, t \geq 0, \tag{1.1}$$

is the abstract form of the flow property: propagating for time s and then for time t is propagating once for $t+s$. Strong continuity is imposed only at the origin; together with the functional equation it forces continuity of $t \mapsto \mathscr{T}(t)x$ on all of $[0, \infty)$ [2].

Definition 1.2 (Infinitesimal generator)**.** The *infinitesimal generator* $\mathscr{A}$ of a C_0-semigroup $\{\mathscr{T}(t)\}_{t\geq 0}$ is the operator

$$\mathscr{A}x = \lim_{t\downarrow 0} \frac{\mathscr{T}(t)x - x}{t}, \tag{1.2}$$

defined on the domain $D(\mathscr{A})$ of those $x \in X$ for which the limit exists in the norm of X. The generator is closed and densely defined.

The generator is the object the rest of the book computes with: it is the datum of the abstract Cauchy problem, and the resolvent, the semigroup, and every functional-calculus operator are built from it. The two structural facts — that $\mathscr{A}$ is closed and that $D(\mathscr{A})$ is dense — are what permit the Laplace-transform machinery of the next section to close.

Theorem 1.3 (The semigroup solves the abstract Cauchy problem)**.** Let $\mathscr{A}$ generate the C_0-semigroup $\{\mathscr{T}(t)\}$. For every $x \in D(\mathscr{A})$ the orbit $u(t) = \mathscr{T}(t)x$ is continuously differentiable on $[0, \infty)$, remains in $D(\mathscr{A})$ for all t, and satisfies

$$\frac{d}{dt}u(t) = \mathscr{A}u(t) = \mathscr{T}(t)\mathscr{A}x, \qquad u(0) = x. \tag{1.3}$$

Thus the semigroup is the solution operator of the autonomous evolution equation.

Proof. Fix $x \in D(\mathscr{A})$ and $t \geq 0$. For $h > 0$,

$$\frac{\mathscr{T}(t+h)x - \mathscr{T}(t)x}{h} = \mathscr{T}(t)\,\frac{\mathscr{T}(h)x - x}{h} \longrightarrow \mathscr{T}(t)\mathscr{A}x \quad (h \downarrow 0), \tag{1.4}$$

using boundedness of $\mathscr{T}(t)$ and the definition of $\mathscr{A}$; the right derivative of u exists and equals $\mathscr{T}(t)\mathscr{A}x$. For the left derivative at $t > 0$, write

$$\begin{aligned} &\frac{\mathscr{T}(t)x - \mathscr{T}(t-h)x}{h} - \mathscr{T}(t)\mathscr{A}x \\ &\quad = \mathscr{T}(t-h)\left(\frac{\mathscr{T}(h)x - x}{h} - \mathscr{A}x\right) + (\mathscr{T}(t-h) - \mathscr{T}(t))\mathscr{A}x. \end{aligned} \tag{1.5}$$

The first term tends to 0 because $\|\mathscr{T}(t-h)\|$ is bounded on the compact interval and $(\mathscr{T}(h)x - x)/h \to \mathscr{A}x$; the second tends to 0 by strong continuity. Hence u is differentiable with $\frac{d}{dt}u(t) = \mathscr{T}(t)\mathscr{A}x$, a continuous function of t, so $u \in C^1$. Finally $\mathscr{T}(t)x \in D(\mathscr{A})$ with $\mathscr{A}\mathscr{T}(t)x = \mathscr{T}(t)\mathscr{A}x$, because $\lim_{h\downarrow 0} h^{-1}(\mathscr{T}(h) - I)\mathscr{T}(t)x = \mathscr{T}(t)\mathscr{A}x$ exists; so $u(t) \in D(\mathscr{A})$ and $\frac{d}{dt}u(t) = \mathscr{A}u(t)$. □

Proposition 1.4 (Uniform growth bound)**.** Every C_0-semigroup satisfies an exponential bound $\|\mathscr{T}(t)\| \leq Me^{\omega t}$ for some $M \geq 1$ and $\omega \in \mathbb{R}$. The infimum of admissible ω is the *growth bound* ω_0.

Proof. By strong continuity and the uniform boundedness principle, $\sup_{0\leq t\leq 1} \|\mathscr{T}(t)\| =: M < \infty$; note $M \geq \|\mathscr{T}(0)\| = 1$. For arbitrary $t \geq 0$ write $t = n + r$ with $n \in \mathbb{N}_0$ and $r \in [0, 1)$. The functional equation gives $\mathscr{T}(t) = \mathscr{T}(1)^n\mathscr{T}(r)$, hence

Chapter 3

The Dunford–Riesz Functional Calculus

Chapter 2 built one function of the operator — the exponential — as a contour integral of the resolvent, and used a second, the fractional power, without saying in what algebra the two live. This chapter supplies the algebra. The Dunford–Riesz functional calculus assigns to each admissible holomorphic function f an operator $f(\mathscr{A})$ by the same contour integral, and does so as a homomorphism: sums go to sums, products to products, the constant 1 to the identity. For a bounded operator the construction is classical; the work is extending it to the unbounded sectorial generators the book cares about, where the spectrum runs to infinity and the naive contour integral diverges. Regularization repairs this, producing the natural functional calculus, and the decisive regularity property — a bound of $f(\mathscr{A})$ by the supremum of f — is the bounded H^∞ calculus, the Hilbert-space gateway to the maximal regularity of Chapter 4.

The chapter delivers the calculus in six steps: the Riesz–Dunford calculus for bounded operators and its spectral mapping theorem (§3.1), the regularized natural calculus for sectorial operators (§3.2), fractional and imaginary powers as its first nontrivial values (§3.3), the bounded H^∞ calculus and its square-function characterization (§3.4), the separation of operators that possess a bounded calculus from those that do not (§3.5), and the identification of the semigroup as the exponential inside the calculus (§3.6). The running example remains the heat generator, whose imaginary powers, calculus bound, and exponential are all computed and checked against the contour quadrature of Chapter 1.

3.1 The Riesz-Dunford Calculus for Bounded Operators

Let $\mathscr{A} \in \mathcal{B}(X)$ be a bounded operator on a Banach space, with spectrum $\sigma(\mathscr{A})$ a nonempty compact subset of $\mathbb{C}$. The classical calculus integrates a holomorphic function against the resolvent over a contour enclosing that spectrum. Its algebraic properties — that the assignment $f \mapsto f(\mathscr{A})$ respects sums, products, and the spectral map — are the template every later extension must preserve, so we establish them with care.

Definition 3.1 (Riesz–Dunford calculus)**.** For $\mathscr{A} \in \mathcal{B}(X)$ and f holomorphic on an open neighbourhood $U \supset \sigma(\mathscr{A})$, define

$$f(\mathscr{A}) = \frac{1}{2\pi i} \int_{\Gamma} f(z)\, \mathscr{R}(z, \mathscr{A})\, dz, \tag{3.1}$$

where $\Gamma \subset U$ is a cycle winding once around $\sigma(\mathscr{A})$ and not at all around $\mathbb{C} \setminus U$.

The integral converges because $z \mapsto f(z)\mathscr{R}(z, \mathscr{A})$ is continuous, hence bounded, on the compact contour Γ. That the definition does not depend on which admissible Γ we choose is the first thing to check, and it is pure Cauchy theory: two admissible cycles are homologous in $U \setminus \sigma(\mathscr{A})$, where the integrand is holomorphic. With well-definedness in hand, the homomorphism property is the substance.

Theorem 3.2 (Homomorphism and spectral mapping)**.** The map $f \mapsto f(\mathscr{A})$ is a unital algebra homomorphism from the holomorphic germs on $\sigma(\mathscr{A})$ into $\mathcal{B}(X)$:

$$(\alpha f + \beta g)(\mathscr{A}) = \alpha f(\mathscr{A}) + \beta g(\mathscr{A}), \quad (fg)(\mathscr{A}) = f(\mathscr{A})g(\mathscr{A}), \quad 1(\mathscr{A}) = I, \tag{3.2}$$

and $\sigma(f(\mathscr{A})) = f(\sigma(\mathscr{A}))$.

Proof. Linearity is immediate. For multiplicativity, take contours Γ_1 inside Γ_2, both around $\sigma(\mathscr{A})$; then

$$f(\mathscr{A})g(\mathscr{A}) = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(z)g(w)\, \mathscr{R}(z, \mathscr{A})\mathscr{R}(w, \mathscr{A})\, dw\, dz. \tag{3.3}$$

Apply the resolvent identity $\mathscr{R}(z, \mathscr{A})\mathscr{R}(w, \mathscr{A}) = \dfrac{\mathscr{R}(z, \mathscr{A}) - \mathscr{R}(w, \mathscr{A})}{w - z}$ and Fubini. The inner w-integral splits into two pieces,

$$\frac{1}{2\pi i}\int_{\Gamma_2}\frac{g(w)}{w-z}\,dw = g(z), \qquad \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(z)}{w-z}\,dz = 0 \quad (w \text{ outside } \Gamma_1), \tag{3.4}$$

the first by Cauchy's formula for z inside Γ_2, the second because $f(z)/(w-z)$ is holomorphic inside Γ_1. What survives is

$$f(\mathscr{A})g(\mathscr{A}) = \frac{1}{2\pi i}\int_{\Gamma_1} f(z)g(z)\,\mathscr{R}(z,\mathscr{A})\,dz = (fg)(\mathscr{A}). \tag{3.5}$$

For spectral mapping, if $\mu \notin f(\sigma(\mathscr{A}))$ then $g(z) = (f(z)-\mu)^{-1}$ is holomorphic near $\sigma(\mathscr{A})$ and $g(\mathscr{A})(f(\mathscr{A})-\mu) = (g\cdot(f-\mu))(\mathscr{A}) = 1(\mathscr{A}) = I$, so $\mu \in \rho(f(\mathscr{A}))$. Conversely, for $\lambda \in \sigma(\mathscr{A})$ factor $f(z)-f(\lambda) = (z-\lambda)h_\lambda(z)$ with h_λ holomorphic; then $f(\mathscr{A})-f(\lambda) = (\mathscr{A}-\lambda)h_\lambda(\mathscr{A})$ cannot be invertible, since $\mathscr{A}-\lambda$ is not, so $f(\lambda) \in \sigma(f(\mathscr{A}))$. The two inclusions give $\sigma(f(\mathscr{A})) = f(\sigma(\mathscr{A}))$ [26]. □

The spectral mapping theorem is the payoff: it says the calculus transports the spectrum by f, so questions about $f(\mathscr{A})$ reduce to questions about f on $\sigma(\mathscr{A})$. Consistency with the algebraic operations one already knows is the next check, and it also pins down the value on rational functions, which is what makes the resolvent itself a value of the calculus.

Proposition 3.3 (Consistency and contour independence)**.** The calculus agrees with the algebraic one on polynomials and on rational functions with poles off $\sigma(\mathscr{A})$: $p(\mathscr{A}) = \sum a_k \mathscr{A}^k$ for $p(z) = \sum a_k z^k$, and $(z-\mu)^{-1} \mapsto \mathscr{R}(\mu,\mathscr{A})$. The value $f(\mathscr{A})$ is independent of the admissible contour.

Proof. For $f(z) = z^k$, deform Γ to a circle $|z| = r > \|\mathscr{A}\|$ and expand the Neumann series $\mathscr{R}(z,\mathscr{A}) = \sum_{n\geq 0} \mathscr{A}^n z^{-n-1}$; term-by-term integration and the residue

$$\frac{1}{2\pi i}\int_{|z|=r} z^{k-n-1}\,dz = \delta_{n,k} \tag{3.6}$$

leave $z^k(\mathscr{A}) = \mathscr{A}^k$. Linearity extends this to all polynomials. For $f(z) = (z-\mu)^{-1}$ with $\mu \notin \sigma(\mathscr{A})$, Cauchy's formula for the operator-valued $\mathscr{R}(\cdot,\mathscr{A})$ gives $\frac{1}{2\pi i}\int_\Gamma \frac{\mathscr{R}(z,\mathscr{A})}{z-\mu}\,dz = \mathscr{R}(\mu,\mathscr{A})$. Contour independence is Cauchy's theorem in $U \setminus \sigma(\mathscr{A})$, where $f(z)\mathscr{R}(z,\mathscr{A})$ is holomorphic [27]. □

A single, decisive application of the calculus is the decomposition of an operator along a spectral splitting, through the projections cut out by locally constant functions.

The standard theory of evolution equations puts the semigroup — the exponential of the generator — at its centre and treats the resolvent, the inverse of the shifted generator, as scaffolding. This book reverses the order. The resolvent is the Laplace transform of the semigroup: prior to it, and, unlike it, available even where no semigroup exists.

That is not a matter of taste. There is a ladder of problems on which the semigroup degrades or vanishes while the resolvent still carries the solution:

• the spectrum betrays you — non-self-adjoint generators, where pseudospectra tell the truth the eigenvalues cannot;

• time refuses to factor — non-autonomous and fractional-in-time problems, where the propagator is built from frozen resolvents;

• nonlinearity — where the nonlinear resolvent solves the Hamilton–Jacobi–Bellman equation.

The book carries one thesis across all three, to the doubly-hard summit: a time-inhomogeneous control problem — correcting circadian misalignment by timed light — solved as a convergent product of frozen-time nonlinear resolvents. Rigorous throughout and exact about where the method stops, it includes the Python code that reproduces every table and figure.