



The value of re-entraining a circadian clock, computed by the resolvent method where no semigroup exists.

# The Resolvent Method for PDEs

*Solving Evolutionary Equations Where the Semigroup  
Cannot*

T. Zamrik



ODIN PRESS



## The Resolvent Method for PDEs

Copyright © 2026 by **T. Zamrik**.

All rights reserved.

No part of this publication may be reproduced, stored in a retrieval system, or transmitted in any form or by any means—electronic, mechanical, photocopying, recording, or otherwise—without the prior written permission of the publisher, except in the case of brief quotations embodied in critical reviews and certain other noncommercial uses permitted by copyright law.

Paperback version, First edition: 2026.

ISBN: forthcoming (Bowker allocation pending)

Trim Size: 6 × 9 inch. Paper color: White.

*Source-code copyright.* All Python source code listings reproduced in this volume—including those in the Appendix, the chapter bodies, and any auxiliary file distributed alongside this book—are the copyrighted intellectual property of **T. Zamrik**. No reproduction, redistribution, adaptation, translation into another programming language, or incorporation into any derivative work, whether academic, commercial, or otherwise, is permitted without the prior written permission of the copyright holder. Single-machine execution by the individual purchaser of this volume for the sole purpose of reproducing a figure or table from this book is granted as a limited, non-transferable, non-sublicensable use and does not constitute redistribution.

Published by Odin Press.

<https://odinpress.org>

This book was typeset in L<sup>A</sup>T<sub>E</sub>X by Odin Press from a manuscript submitted by the author.

Printed and Bound in the UNITED STATES OF AMERICA.

9 8 7 6 5 4 3 2 1 0

# Preface

The standard course in evolution equations puts the semigroup at the centre — the exponential of the generator, the operator that carries an initial state forward in time — and treats the resolvent, the inverse of the shifted generator at a single complex frequency, as the scaffolding one uses to build it. I hold the opposite view, and this book is the argument for it. The resolvent is the Laplace transform of the semigroup: prior to it, in the exact sense that the semigroup is recovered from the resolvent by an inverse transform — a contour quadrature carried to machine precision. Where both objects exist, this is a matter of emphasis. I insist on it because the two do not always both exist, and it is the resolvent, not the semigroup, that survives.

That the semigroup can fail is no pathology. There is a ladder of problems, each weakening one structural assumption, on which the semigroup degrades or ceases to exist while the resolvent still carries the solution. Drop self-adjointness and the eigenfunction expansion need not converge — I once watched a natural spectral series fail to represent a solution whose evolution the resolvent contour then recovered exactly. Drop autonomy and there is no exponential operator to name; the correct object is an evolution family, built from the frozen-time resolvents. Let memory enter and one has a resolvent family with no law of exponents behind it. Drop linearity and the resolvent becomes nonlinear, and the Crandall–Liggett formula generates a nonlinear semigroup from it with no smoothness assumed. The literature treats these cases well, each in its own monograph; what it does not do is carry the single thesis — the resolvent is the primary object — across all of them, to where two assumptions fail at once.

The thesis holds by exhaustion of the ladder, the object at every rung the same one wearing less structure. Part I establishes the Laplace–Bromwich duality and evaluates the analytic semigroup of a sectorial operator by contour quadrature, then builds the Dunford–Riesz functional

---

calculus and the maximal-regularity estimate — both statements about the resolvent alone. Part II removes self-adjointness: where the eigenvalues mislead, the pseudospectrum is the honest diagnostic, and the resolvent contour recovers what the eigenvectors cannot. Part III removes density, autonomy, and the Markov property: the resolvent survives a non-dense domain, the Acquistapace–Terreni construction assembles the propagator from frozen resolvents, and the Mittag-Leffler resolvent family solves the fractional-in-time equation where no semigroup exists at all. Part IV removes linearity: the nonlinear resolvent generates the nonlinear semigroup and solves the Hamilton–Jacobi–Bellman equation as a tower of linear resolvents. The final chapter removes two assumptions at once — nonlinear and non-autonomous — and finds the value function of a real time-inhomogeneous control problem, the timed-light correction of circadian misalignment, as a convergent product of frozen-time nonlinear resolvents, computed on measured data rather than asserted.

I have tried to be as exact about where the method stops as about where it works. The resolvent method is a parabolic phenomenon: it rests on the resolvent decaying in a sector, so that the Bromwich contour can be bent into the left half-plane and the quadrature made to converge. For a genuinely hyperbolic problem — a skew-adjoint generator, a wave equation with time-varying speed — the resolvent decays in no sector, the contour will not bend, and no product of resolvents converges. I reached for the contour out of habit on such a problem once and watched the quadrature refuse; that boundary is not a gap in the method but a theorem about it, and I name it wherever it presses against the construction.

Every numerical result was produced by the code printed in Appendix A and is reproducible from it; where a computed quantity contradicted the claim I had intended to make, I changed the claim. I report each estimate with its constant and each method with its boundary, and I have given nothing in these pages that I have not carried to its last line.

— T. Zamrik, 2026

# Contents

<b>Preface</b>	<b>vii</b>
<b>I The Duality and the Calculus</b>	<b>1</b>
<b>1 The Resolvent and the Semigroup: A Laplace Duality</b>	<b>3</b>
1.1 Strongly Continuous Semigroups and Their Generators . . . . .	4
1.2 The Resolvent as the Laplace Transform of the Semigroup . . . . .	6
1.3 The Hille-Yosida Characterization . . . . .	9
1.4 The Inverse Transform: The Bromwich Integral and Analytic Semigroups . . . . .	10
1.5 The Four Regimes and the Boundary of the Method . . . . .	13
1.6 Exactness in Time and Its Cost . . . . .	17
<b>2 Sectorial Operators and Holomorphic Semigroups</b>	<b>23</b>
2.1 Sectoriality and Its Equivalent Characterizations . . . . .	24
2.2 Generation of Bounded Analytic Semigroups . . . . .	28
2.3 The Hankel Contour and Semigroup Estimates . . . . .	32
2.4 Fractional Powers of Sectorial Operators . . . . .	34
2.5 Interpolation and the Domain Scale . . . . .	38
2.6 Perturbation of Sectorial Generators . . . . .	40
<b>3 The Dunford–Riesz Functional Calculus</b>	<b>45</b>
3.1 The Riesz-Dunford Calculus for Bounded Operators . . . . .	46
3.2 The Natural Calculus for Sectorial Operators . . . . .	49
3.3 Fractional and Imaginary Powers Inside the Calculus . . . . .	53
3.4 The Bounded H-infinity Calculus . . . . .	56
3.5 Operators With and Without a Bounded Calculus . . . . .	59
3.6 The Functional Calculus and the Semigroup . . . . .	60

<b>4</b>	<b>Resolvent Estimates and Maximal Regularity</b>	<b>65</b>
4.1	Resolvent Estimates as the Quantitative Core . . . . .	66
4.2	R-boundedness of Operator Families . . . . .	68
4.3	Operator-Valued Fourier Multipliers . . . . .	70
4.4	The Maximal Regularity Theorem . . . . .	72
4.5	The Resolvent Contour Quadrature . . . . .	75
4.6	Consequences: Well-Posedness and A-Priori Estimates . . . . .	82
<b>II</b>	<b>First Necessity: The Spectrum Betrays You</b>	<b>85</b>
<b>5</b>	<b>Non-Self-Adjoint Generators and the Failure of Eigenexpansion</b>	<b>87</b>
5.1	Loss of the Spectral Theorem . . . . .	88
5.2	Eigenfunctions That Do Not Form a Basis . . . . .	90
5.3	Spectral Projections and Their Norms . . . . .	92
5.4	Non-Normality and the Resolvent Norm . . . . .	94
5.5	When Eigenexpansions Converge and When They Fail . . . . .	97
5.6	The Resolvent Contour as the Correct Object . . . . .	99
<b>6</b>	<b>Pseudospectra: When Eigenvalues Mislead</b>	<b>103</b>
6.1	The Epsilon-Pseudospectrum: Three Equivalent Definitions	104
6.2	Computing and Reading Pseudospectral Portraits . . . . .	106
6.3	The Kreiss Matrix Theorem and Transient Growth . . . . .	112
6.4	Bounds on the Semigroup from Pseudospectra . . . . .	116
6.5	Normal versus Non-Normal . . . . .	119
6.6	The Resolvent Norm as the Honest Diagnostic . . . . .	121
<b>7</b>	<b>Degenerate and Non-Densely-Defined Generators</b>	<b>125</b>
7.1	Non-Densely-Defined Operators and the Failure of Strong Continuity . . . . .	126
7.2	The Da Prato-Sinestrari Theory . . . . .	128
7.3	Intermediate and Extrapolation Spaces . . . . .	130
7.4	Boundary-Degenerate Operators . . . . .	133
7.5	Integrated and Regularized Semigroups . . . . .	136
7.6	The Resolvent as the Surviving Object . . . . .	138

**III Second Necessity: Time Refuses to Factor 143**

**8 Evolution Families and the Nonautonomous Cauchy Problem 145**

8.1	The Nonautonomous Cauchy Problem and the Death of the Semigroup . . . . .	146
8.2	Evolution Families and Propagators . . . . .	149
8.3	Well-Posedness and the Evolution Family . . . . .	152
8.4	The Parabolic and Hyperbolic Dichotomy . . . . .	154
8.5	Exponential Dichotomy and Stability . . . . .	156
8.6	The Frozen Operators and Their Resolvents . . . . .	159

**9 Constructing the Propagator from Frozen Resolvents 163**

9.1	The Construction Problem . . . . .	164
9.2	The Acquistapace-Terreni Conditions . . . . .	166
9.3	The Parabolic Fundamental Solution . . . . .	168
9.4	Regularity and Maximal Regularity . . . . .	172
9.5	The Kato-Tanabe Approach . . . . .	174
9.6	The Boundary: Hyperbolic Failure . . . . .	176

**10 Volterra and Fractional-in-Time Equations: Resolvent Families 181**

10.1	Volterra Equations and Resolvent Families . . . . .	182
10.2	The Subordination Principle . . . . .	184
10.3	Fractional-in-Time Derivatives . . . . .	187
10.4	The Mittag-Leffler Solution Operator . . . . .	189
10.5	Regularity and Resolvent-Family Estimates . . . . .	192
10.6	The Resolvent Family as the Only Surviving Object . . . . .	195

**IV Third Necessity: Nonlinearity and the HJB Equation 199**

**11 Accretive Operators and the Nonlinear Resolvent 201**

11.1	Accretive and $m$ -Accretive Operators . . . . .	202
11.2	The Nonlinear Resolvent . . . . .	204
11.3	The Nonlinear Yosida Approximation . . . . .	207
11.4	The Crandall-Liggett Exponential Formula . . . . .	209
11.5	The Nonlinear Semigroup . . . . .	212
11.6	The Nonlinear Resolvent as the Atom . . . . .	215

<b>12</b>	<b>Viscosity Solutions and the Hamilton-Jacobi-Bellman Equation</b>	<b>219</b>
12.1	Stochastic Control and the HJB Equation . . . . .	220
12.2	The Notion of Viscosity Solutions . . . . .	223
12.3	Comparison and Uniqueness . . . . .	226
12.4	Existence via Perron’s Method . . . . .	228
12.5	The Discounted HJB as a Nonlinear Resolvent . . . . .	230
12.6	Existence and Uniqueness for the Resolvent-HJB . . . . .	232
<b>13</b>	<b>Policy Iteration as Iterated Linear Resolvents</b>	<b>237</b>
13.1	Howard’s Policy Iteration . . . . .	238
13.2	Policy Evaluation as a Linear Resolvent Solve . . . . .	240
13.3	Policy Improvement and the Newton Interpretation . . . . .	243
13.4	Convergence of Policy Iteration . . . . .	245
13.5	The Value Function as an Envelope of Linear Resolvents . . . . .	248
13.6	Practical Realization by Resolvent Solves . . . . .	250
<b>14</b>	<b>Time-Inhomogeneous HJB via the Nonautonomous Nonlinear Resolvent</b>	<b>255</b>
14.1	The Time-Inhomogeneous Control Problem . . . . .	256
14.2	The Doubly-Hard Case . . . . .	258
14.3	Nonautonomous Nonlinear Semigroups . . . . .	260
14.4	The Frozen-Time Nonlinear Resolvents and Their Product . . . . .	262
14.5	Existence and the Convergent Resolvent-Product Representation . . . . .	265
14.6	Computation: The Seasonal HJB . . . . .	267
<b>A</b>	<b>Source Code</b>	<b>275</b>
A.1	Chapter 1 — The Resolvent and the Semigroup: A Laplace Duality . . . . .	275
A.2	Chapter 2 — Sectorial Operators and Holomorphic Semigroups . . . . .	280
A.3	Chapter 3 — The Dunford–Riesz Functional Calculus . . . . .	282
A.4	Chapter 4 — Resolvent Estimates and Maximal Regularity . . . . .	286
A.5	Chapter 5 — Non-Self-Adjoint Generators and the Failure of Eigenexpansion . . . . .	291
A.6	Chapter 6 — Pseudospectra: When Eigenvalues Mislead . . . . .	294
A.7	Chapter 7 — Degenerate and Non-Densely-Defined Generators . . . . .	298

A.8	Chapter 8 — Evolution Families and the Nonautonomous Cauchy Problem . . . . .	300
A.9	Chapter 9 — Constructing the Propagator from Frozen Resolvents . . . . .	303
A.10	Chapter 10 — Volterra and Fractional-in-Time Equations: Resolvent Families . . . . .	306
A.11	Chapter 11 — Accretive Operators and the Nonlinear Resolvent . . . . .	308
A.12	Chapter 12 — Viscosity Solutions and the Hamilton–Jacobi–Bellman Equation . . . . .	311
A.13	Chapter 13 — Policy Iteration as Iterated Linear Resolvents	313
A.14	Chapter 14 — Time-Inhomogeneous HJB via the Nonautonomous Nonlinear Resolvent . . . . .	316
	<b>Bibliography</b>	<b>325</b>
	<b>Index</b>	<b>339</b>
	<b>About the Author</b>	<b>343</b>

# Chapter 1

## The Resolvent and the Semigroup: A Laplace Duality

Two objects compete for primacy in the theory of linear evolution equations. The first is the semigroup  $e^{t\mathcal{A}}$ , the solution operator that advances an initial state forward in time. The second is the resolvent  $(\lambda - \mathcal{A})^{-1}$ , the inverse of a shifted generator at a single complex frequency. Most expositions treat the semigroup as fundamental and the resolvent as an auxiliary computed from it. This book takes the opposite view, and this chapter states the reason. For spectral parameters to the right of the growth bound the resolvent is exactly the Laplace transform of the semigroup, and where the generator is sectorial the semigroup is recovered from the resolvent by a contour integral that a handful of resolvent solves evaluate to near-machine accuracy. The correspondence runs both ways, but it is not symmetric in what it demands: the semigroup requires an autonomous, well-posed evolution before it can be written down, while the resolvent requires only that  $\lambda - \mathcal{A}$  be invertible. On the ladder of harder problems the later chapters climb — non-self-adjoint, non-autonomous, fractional-in-time, fully nonlinear — the semigroup degrades or ceases to exist while the resolvent survives.

The chapter delivers the two-way Laplace–Bromwich correspondence and a four-regime map that fixes, from the outset, where exact-in-time propagation holds and where the method’s advantage ends. The first section introduces the semigroup and its generator; the second identifies the

resolvent as the forward Laplace transform; the third closes the forward direction with the Hille–Yosida characterization; the fourth inverts the transform through the Bromwich integral for sectorial generators; the fifth organizes the landscape into four regimes and marks the hyperbolic boundary plainly; and the last states the thesis the rest of the book generalizes — that zero error in time is realized by the resolvent, not by a marching scheme.

## 1.1 Strongly Continuous Semigroups and Their Generators

Let  $X$  be a Banach space and consider the autonomous Cauchy problem  $\frac{du}{dt} = \mathcal{A}u$ ,  $u(0) = x$ . Its solution operator, when it exists, is a family of bounded operators indexed by time that composes additively; this is the algebraic content of a semigroup, and strong continuity is the minimal regularity that makes the family a genuine evolution [1].

**Definition 1.1** (Strongly continuous semigroup). A *strongly continuous* (or  $C_0$ ) semigroup on a Banach space  $X$  is a family  $\{\mathcal{T}(t)\}_{t \geq 0}$  of bounded linear operators on  $X$  satisfying  $\mathcal{T}(0) = I$ , the functional equation  $\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s)$  for all  $s, t \geq 0$ , and the orbit continuity  $\lim_{t \downarrow 0} \mathcal{T}(t)x = x$  for every  $x \in X$ .

The functional equation

$$\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s), \quad s, t \geq 0, \quad (1.1)$$

is the abstract form of the flow property: propagating for time  $s$  and then for time  $t$  is propagating once for  $t+s$ . Strong continuity is imposed only at the origin; together with the functional equation it forces continuity of  $t \mapsto \mathcal{T}(t)x$  on all of  $[0, \infty)$  [2].

**Definition 1.2** (Infinitesimal generator). The *infinitesimal generator*  $\mathcal{A}$  of a  $C_0$ -semigroup  $\{\mathcal{T}(t)\}_{t \geq 0}$  is the operator

$$\mathcal{A}x = \lim_{t \downarrow 0} \frac{\mathcal{T}(t)x - x}{t}, \quad (1.2)$$

defined on the domain  $D(\mathcal{A})$  of those  $x \in X$  for which the limit exists in the norm of  $X$ . The generator is closed and densely defined.

The generator is the object the rest of the book computes with: it is the datum of the abstract Cauchy problem, and the resolvent, the semigroup,

and every functional-calculus operator are built from it. The two structural facts — that  $\mathcal{A}$  is closed and that  $D(\mathcal{A})$  is dense — are what permit the Laplace-transform machinery of the next section to close.

**Theorem 1.3** (The semigroup solves the abstract Cauchy problem). Let  $\mathcal{A}$  generate the  $C_0$ -semigroup  $\{\mathcal{T}(t)\}$ . For every  $x \in D(\mathcal{A})$  the orbit  $u(t) = \mathcal{T}(t)x$  is continuously differentiable on  $[0, \infty)$ , remains in  $D(\mathcal{A})$  for all  $t$ , and satisfies

$$\frac{d}{dt}u(t) = \mathcal{A}u(t) = \mathcal{T}(t)\mathcal{A}x, \quad u(0) = x. \quad (1.3)$$

Thus the semigroup is the solution operator of the autonomous evolution equation.

*Proof.* Fix  $x \in D(\mathcal{A})$  and  $t \geq 0$ . For  $h > 0$ ,

$$\frac{\mathcal{T}(t+h)x - \mathcal{T}(t)x}{h} = \mathcal{T}(t) \frac{\mathcal{T}(h)x - x}{h} \rightarrow \mathcal{T}(t)\mathcal{A}x \quad (h \downarrow 0), \quad (1.4)$$

using boundedness of  $\mathcal{T}(t)$  and the definition of  $\mathcal{A}$ ; the right derivative of  $u$  exists and equals  $\mathcal{T}(t)\mathcal{A}x$ . For the left derivative at  $t > 0$ , write

$$\begin{aligned} & \frac{\mathcal{T}(t)x - \mathcal{T}(t-h)x}{h} - \mathcal{T}(t)\mathcal{A}x \\ &= \mathcal{T}(t-h) \left( \frac{\mathcal{T}(h)x - x}{h} - \mathcal{A}x \right) + (\mathcal{T}(t-h) - \mathcal{T}(t))\mathcal{A}x. \end{aligned} \quad (1.5)$$

The first term tends to 0 because  $\|\mathcal{T}(t-h)\|$  is bounded on the compact interval and  $(\mathcal{T}(h)x - x)/h \rightarrow \mathcal{A}x$ ; the second tends to 0 by strong continuity. Hence  $u$  is differentiable with  $\frac{d}{dt}u(t) = \mathcal{T}(t)\mathcal{A}x$ , a continuous function of  $t$ , so  $u \in C^1$ . Finally  $\mathcal{T}(t)x \in D(\mathcal{A})$  with  $\mathcal{A}\mathcal{T}(t)x = \mathcal{T}(t)\mathcal{A}x$ , because  $\lim_{h \downarrow 0} h^{-1}(\mathcal{T}(h) - I)\mathcal{T}(t)x = \mathcal{T}(t)\mathcal{A}x$  exists; so  $u(t) \in D(\mathcal{A})$  and  $\frac{d}{dt}u(t) = \mathcal{A}u(t)$ .  $\square$

**Proposition 1.4** (Uniform growth bound). Every  $C_0$ -semigroup satisfies an exponential bound  $\|\mathcal{T}(t)\| \leq Me^{\omega t}$  for some  $M \geq 1$  and  $\omega \in \mathbb{R}$ . The infimum of admissible  $\omega$  is the *growth bound*  $\omega_0$ .

*Proof.* By strong continuity and the uniform boundedness principle,  $\sup_{0 \leq t \leq 1} \|\mathcal{T}(t)\| =: M < \infty$ ; note  $M \geq \|\mathcal{T}(0)\| = 1$ . For arbitrary  $t \geq 0$  write  $t = n + r$  with  $n \in \mathbb{N}_0$  and  $r \in [0, 1)$ . The functional equation gives  $\mathcal{T}(t) = \mathcal{T}(1)^n \mathcal{T}(r)$ , hence

## Chapter 3

# The Dunford–Riesz Functional Calculus

Chapter 2 built one function of the operator — the exponential — as a contour integral of the resolvent, and used a second, the fractional power, without saying in what algebra the two live. This chapter supplies the algebra. The Dunford–Riesz functional calculus assigns to each admissible holomorphic function  $f$  an operator  $f(\mathcal{A})$  by the same contour integral, and does so as a homomorphism: sums go to sums, products to products, the constant 1 to the identity. For a bounded operator the construction is classical; the work is extending it to the unbounded sectorial generators the book cares about, where the spectrum runs to infinity and the naive contour integral diverges. Regularization repairs this, producing the natural functional calculus, and the decisive regularity property — a bound of  $f(\mathcal{A})$  by the supremum of  $f$  — is the bounded  $H^\infty$  calculus, the Hilbert-space gateway to the maximal regularity of Chapter 4.

The chapter delivers the calculus in six steps: the Riesz–Dunford calculus for bounded operators and its spectral mapping theorem (§3.1), the regularized natural calculus for sectorial operators (§3.2), fractional and imaginary powers as its first nontrivial values (§3.3), the bounded  $H^\infty$  calculus and its square-function characterization (§3.4), the separation of operators that possess a bounded calculus from those that do not (§3.5), and the identification of the semigroup as the exponential inside the calculus (§3.6). The running example remains the heat generator, whose imaginary powers, calculus bound, and exponential are all computed and checked against the contour quadrature of Chapter 1.

### 3.1 The Riesz-Dunford Calculus for Bounded Operators

Let  $\mathcal{A} \in \mathcal{B}(X)$  be a bounded operator on a Banach space, with spectrum  $\sigma(\mathcal{A})$  a nonempty compact subset of  $\mathbb{C}$ . The classical calculus integrates a holomorphic function against the resolvent over a contour enclosing that spectrum. Its algebraic properties — that the assignment  $f \mapsto f(\mathcal{A})$  respects sums, products, and the spectral map — are the template every later extension must preserve, so we establish them with care.

**Definition 3.1** (Riesz–Dunford calculus). For  $\mathcal{A} \in \mathcal{B}(X)$  and  $f$  holomorphic on an open neighbourhood  $U \supset \sigma(\mathcal{A})$ , define

$$f(\mathcal{A}) = \frac{1}{2\pi i} \int_{\Gamma} f(z) \mathcal{R}(z, \mathcal{A}) dz, \quad (3.1)$$

where  $\Gamma \subset U$  is a cycle winding once around  $\sigma(\mathcal{A})$  and not at all around  $\mathbb{C} \setminus U$ .

The integral converges because  $z \mapsto f(z)\mathcal{R}(z, \mathcal{A})$  is continuous, hence bounded, on the compact contour  $\Gamma$ . That the definition does not depend on which admissible  $\Gamma$  we choose is the first thing to check, and it is pure Cauchy theory: two admissible cycles are homologous in  $U \setminus \sigma(\mathcal{A})$ , where the integrand is holomorphic. With well-definedness in hand, the homomorphism property is the substance.

**Theorem 3.2** (Homomorphism and spectral mapping). The map  $f \mapsto f(\mathcal{A})$  is a unital algebra homomorphism from the holomorphic germs on  $\sigma(\mathcal{A})$  into  $\mathcal{B}(X)$ :

$$(\alpha f + \beta g)(\mathcal{A}) = \alpha f(\mathcal{A}) + \beta g(\mathcal{A}), \quad (fg)(\mathcal{A}) = f(\mathcal{A})g(\mathcal{A}), \quad 1(\mathcal{A}) = I, \quad (3.2)$$

and  $\sigma(f(\mathcal{A})) = f(\sigma(\mathcal{A}))$ .

*Proof.* Linearity is immediate. For multiplicativity, take contours  $\Gamma_1$  inside  $\Gamma_2$ , both around  $\sigma(\mathcal{A})$ ; then

$$f(\mathcal{A})g(\mathcal{A}) = \frac{1}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} f(z)g(w) \mathcal{R}(z, \mathcal{A})\mathcal{R}(w, \mathcal{A}) dw dz. \quad (3.3)$$

Apply the resolvent identity  $\mathcal{R}(z, \mathcal{A})\mathcal{R}(w, \mathcal{A}) = \frac{\mathcal{R}(z, \mathcal{A}) - \mathcal{R}(w, \mathcal{A})}{w - z}$  and Fubini. The inner  $w$ -integral splits into two pieces,

$$\frac{1}{2\pi i} \int_{\Gamma_2} \frac{g(w)}{w-z} dw = g(z), \quad \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(z)}{w-z} dz = 0 \quad (w \text{ outside } \Gamma_1), \quad (3.4)$$

the first by Cauchy's formula for  $z$  inside  $\Gamma_2$ , the second because  $f(z)/(w-z)$  is holomorphic inside  $\Gamma_1$ . What survives is

$$f(\mathcal{A})g(\mathcal{A}) = \frac{1}{2\pi i} \int_{\Gamma_1} f(z)g(z)\mathcal{R}(z, \mathcal{A}) dz = (fg)(\mathcal{A}). \quad (3.5)$$

For spectral mapping, if  $\mu \notin f(\sigma(\mathcal{A}))$  then  $g(z) = (f(z) - \mu)^{-1}$  is holomorphic near  $\sigma(\mathcal{A})$  and  $g(\mathcal{A})(f(\mathcal{A}) - \mu) = (g \cdot (f - \mu))(\mathcal{A}) = 1(\mathcal{A}) = I$ , so  $\mu \in \rho(f(\mathcal{A}))$ . Conversely, for  $\lambda \in \sigma(\mathcal{A})$  factor  $f(z) - f(\lambda) = (z - \lambda)h_\lambda(z)$  with  $h_\lambda$  holomorphic; then  $f(\mathcal{A}) - f(\lambda) = (\mathcal{A} - \lambda)h_\lambda(\mathcal{A})$  cannot be invertible, since  $\mathcal{A} - \lambda$  is not, so  $f(\lambda) \in \sigma(f(\mathcal{A}))$ . The two inclusions give  $\sigma(f(\mathcal{A})) = f(\sigma(\mathcal{A}))$  [26].  $\square$

The spectral mapping theorem is the payoff: it says the calculus transports the spectrum by  $f$ , so questions about  $f(\mathcal{A})$  reduce to questions about  $f$  on  $\sigma(\mathcal{A})$ . Consistency with the algebraic operations one already knows is the next check, and it also pins down the value on rational functions, which is what makes the resolvent itself a value of the calculus.

**Proposition 3.3** (Consistency and contour independence). The calculus agrees with the algebraic one on polynomials and on rational functions with poles off  $\sigma(\mathcal{A})$ :  $p(\mathcal{A}) = \sum a_k \mathcal{A}^k$  for  $p(z) = \sum a_k z^k$ , and  $(z - \mu)^{-1} \mapsto \mathcal{R}(\mu, \mathcal{A})$ . The value  $f(\mathcal{A})$  is independent of the admissible contour.

*Proof.* For  $f(z) = z^k$ , deform  $\Gamma$  to a circle  $|z| = r > \|\mathcal{A}\|$  and expand the Neumann series  $\mathcal{R}(z, \mathcal{A}) = \sum_{n \geq 0} \mathcal{A}^n z^{-n-1}$ ; term-by-term integration and the residue

$$\frac{1}{2\pi i} \int_{|z|=r} z^{k-n-1} dz = \delta_{n,k} \quad (3.6)$$

leave  $z^k(\mathcal{A}) = \mathcal{A}^k$ . Linearity extends this to all polynomials. For  $f(z) = (z - \mu)^{-1}$  with  $\mu \notin \sigma(\mathcal{A})$ , Cauchy's formula for the operator-valued  $\mathcal{R}(\cdot, \mathcal{A})$  gives  $\frac{1}{2\pi i} \int_{\Gamma} \frac{\mathcal{R}(z, \mathcal{A})}{z - \mu} dz = \mathcal{R}(\mu, \mathcal{A})$ . Contour independence is Cauchy's theorem in  $U \setminus \sigma(\mathcal{A})$ , where  $f(z)\mathcal{R}(z, \mathcal{A})$  is holomorphic [27].  $\square$

A single, decisive application of the calculus is the decomposition of an operator along a spectral splitting, through the projections cut out by locally constant functions.

The standard theory of evolution equations puts the semigroup — the exponential of the generator — at its centre and treats the resolvent, the inverse of the shifted generator, as scaffolding. This book reverses the order. The resolvent is the Laplace transform of the semigroup: prior to it, and, unlike it, available even where no semigroup exists.

That is not a matter of taste. There is a ladder of problems on which the semigroup degrades or vanishes while the resolvent still carries the solution:

- the spectrum betrays you — non-self-adjoint generators, where pseudospectra tell the truth the eigenvalues cannot;
- time refuses to factor — non-autonomous and fractional-in-time problems, where the propagator is built from frozen resolvents;
- nonlinearity — where the nonlinear resolvent solves the Hamilton–Jacobi–Bellman equation.

The book carries one thesis across all three, to the doubly-hard summit: a time-inhomogeneous control problem — correcting circadian misalignment by timed light — solved as a convergent product of frozen-time nonlinear resolvents. Rigorous throughout and exact about where the method stops, it includes the Python code that reproduces every table and figure.

