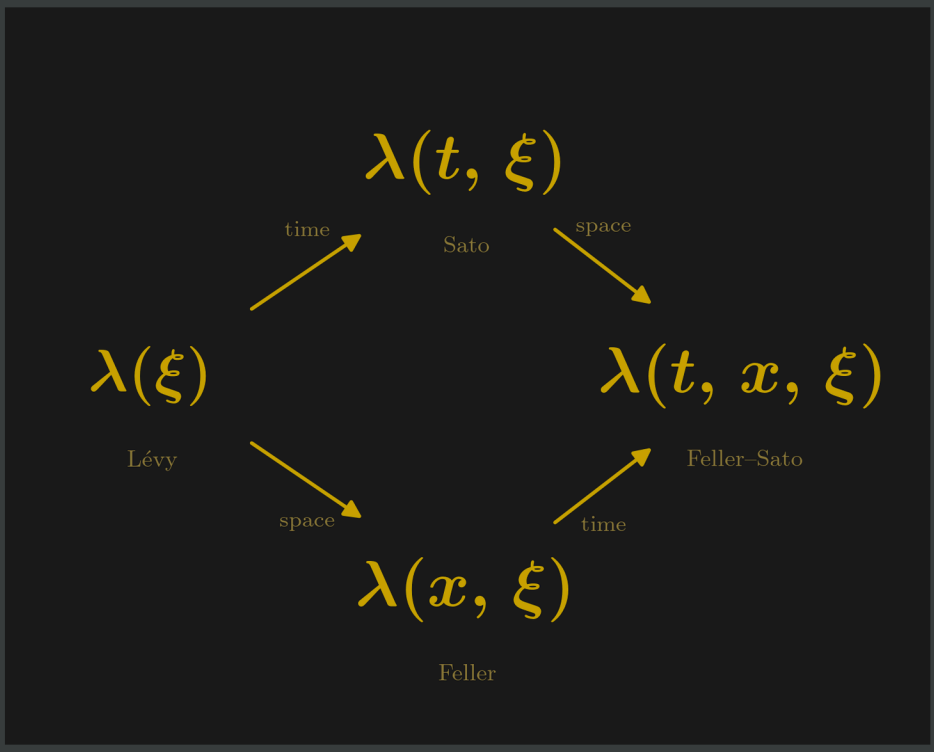


The symbol through four classes, one symmetry broken at each step.

Breaking Homogeneity: From Lévy to Feller–Sato Processes

Pseudodifferential Operators and Markovian Potential Theory

T. Zamrik

ODIN PRESS

Breaking Homogeneity: From Lévy to Feller–Sato Processes

© 2025 **T. Zamrik**. All Rights Reserved.

All rights reserved. No part of this book — including text, equations, figures, and the algorithms and pseudocodes contained herein — may be reproduced, stored in a retrieval system, or transmitted in any form or by any means (electronic, mechanical, photocopying, recording, or otherwise) without the prior written permission of the author, except in the case of brief quotations embodied in critical reviews and certain other noncommercial uses permitted by copyright law. For permission requests, contact the author directly.

The algorithms and pseudocodes presented in this book are original work by the author. It may be used for personal study and non-commercial research purposes, provided that the source is acknowledged. Any other use requires written permission from the author.

Every effort has been made to ensure the accuracy of the mathematical content. The author accepts no liability for errors or omissions.

Published by Odin Press.
`https://odinpress.org`
This book was typeset in LaTeX in Latin Modern Roman by Odin Press from a manuscript submitted by the author.
Paperback version, trim size 6 × 9 inches.
ISBN: forthcoming (Bowker allocation pending)

9 8 7 6 5 4 3 2 1 0

Preface

In 1966, Mark Kac asked whether one can hear the shape of a drum. The question is this: if you listen only to the frequencies at which a drumhead vibrates, can you determine its geometry? The answer, worked out over the following decades, is: almost, but not quite. The spectrum of the Laplacian encodes a great deal about a domain, but not everything.

This book asks the probabilist's version of the same question. Given a Markov process, what does its generator tell you about its trajectories? The generator is a pseudodifferential operator, and its symbol — the function $\lambda(t, x, \xi)$ in the Fourier variable ξ, possibly depending on time t and position x — encodes the infinitesimal structure of the process in the same way that eigenvalues encode the shape of a drum. The symbol is the object this book is about.

The symbol takes four forms, each corresponding to a distinct class of processes. In its simplest form, $\lambda(\xi)$, it is constant in time and space: this is the characteristic exponent of a Lévy process, and the Lévy–Khintchine theorem gives the complete classification. When time-dependence is introduced, the symbol becomes $\lambda(t, \xi)$, and the associated processes are the additive and self-similar processes studied by Sato. When spatial dependence replaces temporal dependence, the symbol becomes $\lambda(x, \xi)$: this is the domain of Feller processes, characterised by Courège's theorem as precisely those Markov processes whose gener-

ators satisfy the positive maximum principle. The full symbol $\lambda(t, x, \xi)$, depending on both time and position, governs the Feller–Sato class — the most general family of jump-driven Markov processes with a rich potential theory.

The four parts of this book develop these four cases in order, each building on the last. Part I treats Lévy processes: Chapter 1 introduces infinite divisibility and the Lévy–Khintchine representation; Chapter 2 establishes the generator as a pseudodifferential operator and develops the theory of Bernstein functions and Bochner subordination; Chapter 3 develops Markovian potential theory — resolvents, excessive functions, capacity, and Hunt's theorem; Chapter 4 is devoted to the Wiener–Hopf factorisation and the ladder subordinators; Chapter 5 treats scale functions for spectrally negative processes. Part II breaks temporal homogeneity: Chapters 6–8 develop additive processes, self-similar processes, and potential theory for Sato processes, showing which results from Part I survive the loss of stationarity and which require new arguments. Part III breaks spatial homogeneity: Chapters 9–12 treat Feller semigroups, Courège's theorem, existence results via the martingale problem and Levi parametrix, and potential theory with variable symbol. Part IV combines both inhomogeneities: Chapters 13–15 develop the full symbol calculus, evolution systems of operators, and the hero result of the book — heat kernel estimates for the Feller–Sato class in the spirit of Knopova, Schilling, and Wang. Chapter 16 closes with potential theory at full generality: the Dirichlet problem for non-local operators, exit distributions, polar sets, and the Wiener criterion.

Three appendices support the main text. Appendix A collects the functional analysis and semigroup theory used throughout. Appendix B develops the symbol calculus of pseudodifferential operators systematically. Appendix C discusses numerical methods for the symbol: quadrature for the Lévy–Khintchine integral, FFT-based discretisation of $\mathrm{Op}[\lambda(x, \xi)]$, and symbol-based heat kernel bounds. The book is self-

contained via these appendices; the prerequisites are undergraduate probability, functional analysis, and real analysis.

The single structural feature that ties all four parts together is the following: for every fixed value of the parameters (t, x), the map $\xi \mapsto \lambda(t, x, \xi)$ is a continuous negative definite function. This property, which originates in the Lévy–Khintchine theorem of Chapter 1, is preserved under all the deformations introduced in Parts II, III, and IV. It is the constraint that makes the symbol the right object to study, and the reason the answer to the probabilist's version of Kac's question is richer and more complete than the analyst's.

T. Zamrik
April 2025

Contents

Preface **vii**

List of Symbols **xxi**

1 Infinite Divisibility and the Lévy–Khintchine Representation **1**

1.1 Introduction 1
1.2 Infinite Divisibility 2
1.3 The Lévy Measure and the Lévy Triplet 3
1.4 The Lévy–Khintchine Theorem 4
1.4.1 The Truncation Function 5
1.5 The Lévy–Itô Decomposition 6
1.6 Continuous Negative Definite Functions 7
1.7 The Symbol $\lambda(\xi)$ 9
1.8 Self-Decomposability 10
1.9 Convergence and Continuity 11
1.10 Examples 12
1.11 Summary and Outlook 14

2 The Generator as a Pseudodifferential Operator **17**

2.1 Introduction 17
2.2 The C_0-Semigroup and Its Generator 18
2.3 The Generator as a Fourier Multiplier 19

2.3.1 Explicit Form via the Lévy–Khintchine Formula 20
2.3.2 Domain of the Generator 21
2.4 The Pseudodifferential Operator Notation 22
2.5 Bernstein Functions and Subordinators 22
2.5.1 Complete Bernstein Functions 23
2.5.2 Examples of Bernstein Functions 24
2.6 Bochner Subordination 25
2.6.1 Examples of Subordination 26
2.7 Preservation of Negative Definiteness 26
2.8 The Generator in Operator Form 27
2.9 Connection to the Spectral Theory of the Generator . 28
2.10 Summary and Outlook 29

3 Markovian Potential Theory: Resolvents, Excessive Functions, Capacity 31
3.1 Introduction 31
3.2 The Resolvent Family 32
3.3 The Resolvent Identity and Generator Inversion 34
3.4 Excessive Functions and the Riesz Decomposition . . . 36
3.5 Potential Kernels and Green's Functions 37
3.6 Capacity and Equilibrium Measures 39
3.7 Hunt's Theorem 40
3.8 Choquet's Theorem and Analytic Capacity 41
3.9 Polar and Semipolar Sets 43
3.10 Outlook: Deforming the Potential Theory 44

4 Ladder Processes and Wiener–Hopf Factorization 47
4.1 Introduction 47
4.2 Running Extrema 48
4.3 Ladder Processes Are Subordinators 49
4.4 Ladder Height Potential Measures 50
4.5 The Wiener–Hopf Factorization 51

4.6 The Resolvent as a Product of Half-Line Resolvents . 53
4.7 First Passage and the Factorization 54
4.8 The Kou Process: Explicit Wiener–Hopf Factorization 55
4.9 Outlook: Why Homogeneity Is Essential 58

5 Scale Functions for Spectrally Negative Lévy Processes 61
5.1 Introduction 61
5.2 Spectrally Negative Lévy Processes 62
5.3 The Laplace Exponent and Its Right-Inverse 63
5.4 The First Scale Function $W^{(q)}$ 64
5.5 The Second Scale Function $Z^{(q)}$ 66
5.6 The q-Resolvent Density 67
5.7 The Two-Sided Exit Problem 68
5.8 Ruin Probability 70
5.9 Connection to the Wiener–Hopf Factorization 71
5.10 The Cramér–Lundberg Process: Explicit Computations 72
5.11 Outlook: Why Scale Functions Break for Inhomogeneous Processes 75

6 Additive Processes, Self-Similarity, and the Time-Dependent Symbol 77
6.1 Introduction 77
6.2 What Breaks and What Survives 79
6.3 Additive Processes 80
6.4 Sato's Characterisation Theorem 81
6.5 Self-Similar Additive Processes 82
6.6 Operator Self-Similarity 84
6.7 The Lamperti Transformation 86
6.8 The Time-Dependent Symbol 87
6.9 The Integral Decomposition 88
6.10 The Self-Similar Scaling Identity 90

6.11 Uniqueness 91
6.12 Contrast with the Spatially-Dependent Symbol 92
6.13 Figure: Lévy Process versus Sato Process 93
6.14 Outlook 93

7 Potential Theory for Sato Processes **95**
7.1 Introduction 95
7.2 The Time-Dependent Resolvent 97
7.3 The Time-Dependent Generator 99
7.4 Dynkin's Formula 100
7.5 Excessive Functions in the Additive Setting 102
7.6 The Time-Dependent Potential Kernel 105
7.7 Capacity for Sato Processes 107
7.8 Polar Sets for Sato Processes 109
7.9 What Breaks: The Absence of a Stationary Potential Theory 111

8 Feller Semigroups and the Positive Maximum Principle **115**
8.1 What Survives from Fluctuation Theory 115
8.2 Feller Semigroups 116
8.3 The Hille–Yosida Theorem and the Yosida Approximation 118
8.4 The Positive Maximum Principle 121
8.5 Explicit Verification of the Positive Maximum Principle 123
8.6 Dynkin's Formula for Feller Processes 125
8.7 The Strong Markov Property 128
8.8 Strong Feller, Ultra-Feller, and the Symbol 129
8.9 Variable-Order Stable Processes 131

9 Courège's Theorem: Characterizing Variable Symbols **133**
9.1 Introduction 133

9.2 The Domain and the Positive Maximum Principle . . 134
9.3 Courège's Theorem 136
9.4 The Converse 139
9.5 Symbol Regularity 140
9.6 The Lévy System 143
9.7 Variable Order 145
9.8 Explicit Examples 146
9.9 The Gap to Chapter 10 149

10 Construction of Feller Processes: Martingale Problem and Parametrix 151
10.1 Introduction 151
10.1.1 Notation and Scope 152
10.2 Standing Assumptions 153
10.3 The Martingale Problem 154
10.3.1 Formulation 154
10.3.2 Well-Posedness 154
10.3.3 Equivalence with the Feller Property 155
10.4 Frozen-Coefficient Approximation 156
10.4.1 The Frozen Lévy Process 156
10.4.2 The Zeroth-Order Parametrix 157
10.4.3 The Defect Estimate 158
10.5 The Volterra Iteration 160
10.5.1 Construction of the Correction Series 160
10.5.2 Convergence of the Series 160
10.5.3 The Fundamental Solution 162
10.6 Verification of the Feller Property 163
10.7 The Euler Scheme Approximation 164
10.8 Uniqueness and Comparison of the Two Constructions 166
10.9 Pointwise Transition Density Bounds 167
10.9.1 Upper Bound and Corrections 167
10.9.2 Hölder Regularity in the Spatial Variable . . . 168

10.9.3 On-Diagonal Asymptotics 169
10.10 Example: Variable-Order Stable Process 170

11 Potential Theory for Feller Processes **173**
11.1 Introduction 173
11.2 Resolvent and Excessive Functions 174
11.2.1 The Feller Resolvent 174
11.2.2 Excessive Functions and the Riesz Decomposition 175
11.3 Capacity and Polar Sets 176
11.3.1 Variable-Symbol Capacity 176
11.3.2 Polar Sets and Regular Points 179
11.4 Green Function Estimates in Bounded Domains 180
11.4.1 The Killed Semigroup and Green Function . . . 180
11.4.2 Two-Sided Green Function Estimates 180
11.5 Recurrence and Transience 182
11.6 The Harnack Inequality 183
11.6.1 Harmonic Functions 183
11.6.2 The Scale-Invariant Harnack Inequality 183
11.7 Dirichlet Form and Energy 185
11.7.1 The Bilinear Form 185
11.7.2 Variational Capacity 186
11.8 Balayage and Harmonic Measure 187
11.8.1 Hitting Times and Balayage 187
11.8.2 The Poisson Kernel 187
11.9 Kato Class and Feynman–Kac Perturbations 189
11.9.1 The Kato Class 189
11.9.2 The Feynman–Kac Semigroup 190
11.10 Summary and Figures 191

12 Time-Inhomogeneous Feller Processes and Evolution Systems **195**
12.1 Introduction 195

12.2 Evolution Systems and the Full Symbol 197
12.2.1 Evolution Systems of Operators 197
12.2.2 Time-Inhomogeneous Feller Processes 199
12.2.3 The Full Symbol 200
12.2.4 Norm Estimates and the Contraction Property 202
12.3 Fluctuation Theory: What Survives and What Breaks 205
12.3.1 What Breaks 205
12.3.2 Partial Survivors 207
12.3.3 Open Problems 209
12.4 The Martingale Problem for $\lambda(t, x, \xi)$ 210
12.4.1 Formulation 210
12.4.2 Existence via the Levi Parametrix 210
12.4.3 Well-Posedness 212
12.5 Parametrix Construction of the Transition Density . . 213
12.5.1 The Frozen Approximation 213
12.5.2 The Volterra Iteration 213
12.5.3 Existence and Positivity 214
12.6 Strong Markov Property and Path Regularity 216
12.6.1 Càdlàg Paths 216
12.6.2 Strong Markov Property 217
12.6.3 Feller Property of the Propagator 218
12.7 Symbol Approximation by Step Functions in t 218
12.7.1 Piecewise-Constant Symbols 219
12.7.2 Strong Convergence of Evolution Systems . . . 219
12.7.3 Measurability Suffices 220
12.8 Concrete Examples 221
12.8.1 Example: Variable-Order Stable Process 221
12.8.2 Example: Blended Lévy Interpolation 223
12.8.3 Example: NIG with Time-Varying Parameters 224
12.9 Summary and Connection to Part IV 225

13 Pseudodifferential Calculus for $\lambda(t, x, \xi)$ **227**
13.1 Motivation and Quantization Schemes 227
13.2 Oscillatory Integrals 230
13.3 Symbol Classes 232
13.4 L^2 Boundedness 234
13.5 L^p Theory 236
13.6 Pseudolocality and Wavefront Sets 237
13.7 Composition 238
13.8 Elliptic Theory and Parametrix 241
13.9 Hypoellipticity 242
13.10 Gårding Inequalities 244
13.11 Commutator Estimates 245
13.12 Cauchy Problem Energy Estimates 246
13.13 Worked Examples 248

14 Heat Kernel Estimates **253**
14.1 Heat Kernel: Existence and Basic Properties 254
14.1.1 Definition and Chapman–Kolmogorov Identity 254
14.1.2 Existence and Continuity 255
14.2 Parametrix-to-Bounds Bridge 258
14.3 Two-Sided Bounds 260
14.3.1 Symbol Profile and On-Diagonal Estimates . . 260
14.3.2 Upper Bound 261
14.3.3 Lower Bound and Two-Sided Comparison . . . 262
14.3.4 Aronson Gaussian Bound 264
14.4 Small-Time Asymptotics 265
14.4.1 On-Diagonal Regime 265
14.4.2 Off-Diagonal Regime 266
14.5 Large-Time Behavior and Recurrence 268
14.5.1 Recurrence and Transience 268
14.6 Gradient Bounds and Parabolic Harnack Inequality . . 269
14.6.1 Gradient Estimates 269

14.6.2 Parabolic Harnack Inequality 270
14.7 Spectral Theory 272
14.7.1 Principal Eigenvalue and Positivity 272
14.7.2 Spectral Gap and Intrinsic Ultracontractivity . 274
14.8 Semigroup Theory from the Kernel 275
14.8.1 Feller Semigroup 275
14.8.2 Generator Domain 277
14.9 Comparison and Monotonicity 278
14.10 Subordination and Kernel Transformation 280
14.11 Connection to Knopova–Schilling–Wang and All Four Process Classes 282

15 Potential Theory at Full Generality **285**
15.1 Introduction 285
15.2 Green's Function 286
15.3 Non-Local Maximum Principle 288
15.4 Riesz Decomposition 289
15.5 The 3G Inequality 291
15.6 Harmonic Functions and the Mean Value Property . . 292
15.7 Poisson Kernel 293
15.8 Polar Sets and Hausdorff Dimension 296
15.9 Elliptic Harnack Inequality 297
15.10 The Dirichlet Problem 298
15.11 Exit Distribution and Capacity 300
15.12 Wiener Criterion 302
15.13 Feynman–Kac Formula and Kato Class 303
15.14 Martin Boundary 305
15.15 Liouville Theorem 307
15.16 Worked Example: Explicit Solution on the Ball 308
15.17 Connection to Variable-Order Fractional PDEs 309

A Elements of Functional Analysis and Semigroup Theory **313**
A.1 Banach and Hilbert Spaces 313
A.2 Bounded Operators and Spectral Theory 315
A.3 C_0-Semigroups 317
A.4 Resolvent as Laplace Transform 319
A.5 Feller Semigroups 320

B Pseudodifferential Operators: Symbol Classes and Calculus **323**
B.1 The Schwartz Space and Fourier Transform 323
B.2 Tempered Distributions 325
B.3 Symbol Classes and Quantisation 326
B.4 Oscillatory Integrals and L^2 Boundedness 327
B.5 Sobolev Mapping, Composition, and Adjoint 329
B.6 Elliptic Symbols and the Parametrix 331

C Numerical Methods for PDO Symbols **333**
C.1 Evaluating the Symbol 334
C.2 Discretizing the Symbol Operator 337
C.3 Non-local Operator Discretization 339
C.4 Symbol Composition 341
C.5 Parametrix Construction 343
C.6 Heat Kernel Computation 344

References **347**

Index **355**

Chapter 1

Infinite Divisibility and the Lévy–Khintchine Representation

1.1 Introduction

The classical central limit theorem identifies a privileged class of distributions — the Gaussian — as the universal attractor under summation and scaling. Behind this universality lies a structural property: the Gaussian distribution can be written as a product of identical factors in the space of characteristic functions for every positive integer n. This chapter develops the precise formulation of that property, extends it to the full class of distributions sharing it, and extracts from them the central analytic object of this book.

The theory of infinite divisibility was developed by Bruno de Finetti in the 1930s and brought to its definitive form by Paul Lévy and Andrei Khintchine. The representation theorem that bears their names, and the pathwise decomposition of the associated processes into independent components due to Lévy and Itô, constitute the foundation on

which everything that follows is built. The standard references are Sato [48], Bertoin [6], Jacob [27], and Applebaum [1]; we follow their conventions throughout, with the emphasis on the characteristic exponent as the primary object.

A single formula emerges at the end of this chapter and will not change form until Chapter 12. It is

$$\lambda(\xi) = ib\xi - \tfrac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{i\xi y} - 1 - i\xi y\,\mathbf{1}_{|y|\le 1}\right)\nu(dy), \tag{1.1}$$

the characteristic exponent of an infinitely divisible distribution. In Parts II, III, and IV this formula will be progressively deformed: the triplet (b, σ^2, ν) will acquire dependence on time, on space, and finally on both simultaneously. But the structure of (1.1) persists throughout as the skeleton of the symbol of a pseudodifferential operator.

1.2 Infinite Divisibility

Definition 1.1 (Infinitely divisible distribution)**.** A probability measure μ on $\mathbb{R}$ is *infinitely divisible* if for every $n \in \mathbb{N}$ there exists a probability measure μ_n on $\mathbb{R}$ such that

$$\mu = \underbrace{\mu_n * \mu_n * \cdots * \mu_n}_{n \text{ factors}}, \tag{1.2}$$

where $*$ denotes convolution.

In terms of characteristic functions, (1.2) is the condition that

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}} e^{i\xi x}\,\mu(dx) = \big(\widehat{\mu_n}(\xi)\big)^n \tag{1.3}$$

for some characteristic function $\widehat{\mu_n}$, for every $n \ge 1$. The Gaussian measure $\mathcal{N}(0,\sigma^2)$ is infinitely divisible: $\widehat{\mu}(\xi) = e^{-\sigma^2\xi^2/2}$ and $(e^{-\sigma^2\xi^2/2})^{1/n} = e^{-\sigma^2\xi^2/(2n)}$ is the characteristic function of $\mathcal{N}(0,\sigma^2/n)$. The Poisson

measure with parameter $\lambda_0 > 0$ and the Cauchy distribution are infinitely divisible. The uniform distribution on $[0, 1]$ is not.

A fundamental structural consequence is that the characteristic function of an infinitely divisible measure never vanishes, which allows the characteristic exponent to be defined globally.

Proposition 1.2 (Non-vanishing of $\widehat{\mu}$)**.** *If μ is infinitely divisible then $\widehat{\mu}(\xi) \neq 0$ for all $\xi \in \mathbb{R}$, and the function $\lambda(\xi) = \log \widehat{\mu}(\xi)$ (principal branch) is a well-defined continuous function.*

Proof. Suppose $\widehat{\mu}(\xi_0) = 0$ for some ξ_0. For each n, we have $\widehat{\mu}(\xi_0) = \widehat{\mu_n}(\xi_0)^n = 0$, so $\widehat{\mu_n}(\xi_0) = 0$. Now use the standard inequality: for any probability measure ν,

$$|\widehat{\nu}(\xi)|^2 \leq \tfrac{1}{2}\big(1 + \operatorname{Re} \widehat{\nu * \tilde{\nu}}(\xi)\big) \tag{1.4}$$

where $\tilde{\nu}$ is the reflection of ν. Applying this to μ_n and tracking the limit $n \to \infty$ shows that $\widehat{\mu}(\xi_0) = 0$ forces all nearby characteristic functions to vanish as well, contradicting continuity of $\widehat{\mu}$ at $\xi_0 = 0$ where $\widehat{\mu}(0) = 1$. The full argument is given in Sato [48] Lemma 7.5. □

Remark 1.3. Proposition 1.1 ensures that $\lambda(\xi) = \log \widehat{\mu}(\xi)$ is well-defined and continuous. The Lévy–Khintchine theorem characterises precisely which continuous functions arise this way.

1.3 The Lévy Measure and the Lévy Triplet

Before stating the main theorem, we identify the analytic ingredients that parametrise all infinitely divisible distributions.

Definition 1.4 (Lévy measure)**.** A σ-finite measure ν on $\mathbb{R} \setminus \{0\}$ is a *Lévy measure* if

$$\int_{\mathbb{R}\setminus\{0\}} \min(1, y^2)\, \nu(dy) < \infty. \tag{1.5}$$

Chapter 3

Markovian Potential Theory: Resolvents, Excessive Functions, Capacity

3.1 Introduction

Classical potential theory begins with harmonic and superharmonic functions, objects defined analytically through the Laplacian. The probabilistic revolution of the mid-twentieth century, carried through by Hunt, Doob, Blumenthal, and Getoor, revealed that these analytic notions have intrinsic probabilistic counterparts: harmonic functions correspond to martingales, superharmonic functions to supermartingales, and the Laplacian itself to the generator of Brownian motion. The monograph of Blumenthal and Getoor [7] systematised this dictionary and extended it to arbitrary Markov processes. For Lévy processes the picture is especially clean: the homogeneous symbol $\lambda(\xi)$ determines the spectral theory of the generator, and consequently every potential-

theoretic object of the process can be computed from $\lambda(\xi)$ via Fourier analysis.

This chapter builds the potential-theoretic apparatus that will be deformed in Parts II–IV of the book when $\lambda(\xi)$ is replaced by a symbol $\lambda(t, x, \xi)$ depending on position and time. We define the resolvent family $\{\mathscr{R}_q\}_{q>0}$, establish the resolvent identity, and show that $\mathscr{R}_q$ inverts the shifted generator $q - \mathscr{L}$. We then introduce q-excessive functions as the probabilistic counterpart of q-superharmonic functions, and prove the Riesz decomposition that factorises every excessive function into a potential part and a harmonic part. The potential kernel U and its density (the Green's function) are constructed via Fourier analysis on $\mathbb{R}^d$. We define capacity through equilibrium measures, establish Hunt's theorem connecting capacity to polarity, prove that capacity is a Choquet capacity, and characterise polar sets and semipolar sets probabilistically.

Blumenthal and Getoor [7], Port and Stone [45], and Chung and Walsh [13] are the primary references for this chapter. Jacob's second volume [28] provides the Fourier-analytic perspective linking the symbol $\lambda(\xi)$ to potential-theoretic quantities. We consistently work with a Lévy process $(X_t)_{t\geq 0}$ on $\mathbb{R}^d$ with characteristic exponent $\lambda(\xi)$ as defined in Chapter 1.

3.2 The Resolvent Family

Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$ with semigroup $\mathcal{T}_t f(x) = \mathbb{E}_x[f(X_t)]$ acting on $\mathcal{B}_b(\mathbb{R}^d)$, the bounded Borel functions. The resolvent family encodes all time-integrated information about the process.

Definition 3.1 (Resolvent Operator)**.** For $q > 0$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$, the *$q$-resolvent* $\mathscr{R}_q$ is the operator

$$\mathscr{R}_q f(x) = \mathbb{E}_x\left[\int_0^\infty e^{-qt} f(X_t)\, dt\right] = \int_0^\infty e^{-qt}\, \mathcal{T}_t f(x)\, dt. \tag{3.1}$$

The resolvent $\mathscr{R}_q f(x)$ represents the expected discounted occupation time of the process starting at x, weighted by the payoff function f, at discount rate q.

Proposition 3.2 (Boundedness and Positivity of the Resolvent)**.** *For every $q > 0$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$:*

(i) $\|\mathscr{R}_q f\|_\infty \le q^{-1}\|f\|_\infty$.

(ii) If $f \ge 0$ then $\mathscr{R}_q f \ge 0$.

(iii) The map $q \mapsto q\mathscr{R}_q f(x)$ is non-increasing for $f \ge 0$, with $\lim_{q\to\infty} q\mathscr{R}_q f(x) = f(x)$ and $\lim_{q\to 0^+} q\mathscr{R}_q f(x) = 0$ when the process is transient.

Proof. Step 1 (Boundedness). By definition and $\|\mathcal{T}_t f\|_\infty \le \|f\|_\infty$,

$$\|\mathscr{R}_q f\|_\infty \le \int_0^\infty e^{-qt}\|\mathcal{T}_t f\|_\infty\, dt \le \|f\|_\infty \int_0^\infty e^{-qt}\, dt = \frac{\|f\|_\infty}{q}. \tag{3.2}$$

Step 2 (Positivity). If $f \ge 0$ then $\mathcal{T}_t f(x) = \mathbb{E}_x[f(X_t)] \ge 0$ for all t and x, so the integral in (A.7) is non-negative.

Step 3 (Monotonicity in q). For $f \ge 0$, $q\mathscr{R}_q f(x) = q\int_0^\infty e^{-qt}\mathcal{T}_t f(x)\, dt$. Differentiating in q: $\partial_q[q\mathscr{R}_q f(x)] = \int_0^\infty e^{-qt}(1 - qt)\mathcal{T}_t f(x)\, dt$, which is not sign-definite. In fact $q \mapsto q\mathscr{R}_q f(x)$ is non-increasing: for $p < q$, $p\mathscr{R}_p f(x) \ge q\mathscr{R}_q f(x)$ follows from the resolvent identity $\mathscr{R}_p - \mathscr{R}_q = (q-p)\mathscr{R}_p\mathscr{R}_q$ and $\mathscr{R}_p, \mathscr{R}_q \ge 0$. □

Remark 3.3. The limit $\lim_{q\to\infty} q\mathscr{R}_q f(x) = f(x)$ holds pointwise for every $f \in C_b(\mathbb{R}^d)$ and every $x \in \mathbb{R}^d$. This is the probabilistic analogue of the approximation-of-identity property of the resolvent in functional analysis; it follows from $\mathcal{T}_0 = \mathrm{Id}$ and the right-continuity of $t \mapsto \mathcal{T}_t f(x)$.

In 1966, Mark Kac asked whether one can hear the shape of a drum. This book asks the probabilist's version: what does the generator of a Markov process tell you about the process itself? The answer is encoded in a single function — the symbol — and this book is its complete theory.

Breaking Homogeneity traces four stages of increasing generality. Each part breaks one more symmetry: temporal homogeneity falls first (Sato processes), then spatial homogeneity (Feller processes), until the fully inhomogeneous Feller–Sato class is reached. At every stage, the symbol remains a negative definite function in the Fourier variable — the single invariant tying the entire theory together.

The book covers infinite divisibility, Wiener–Hopf factorisation, Courège's theorem, the parametrix construction, and heat kernel estimates. Potential theory is developed and compared systematically across all four classes.

Dr. T. Zamrik is an applied mathematician working at the intersection of stochastic analysis, probability theory, stochastic control, and partial differential equations.

ISBN 13 978-147-57-6582-3
Paperback
9 781475 765823