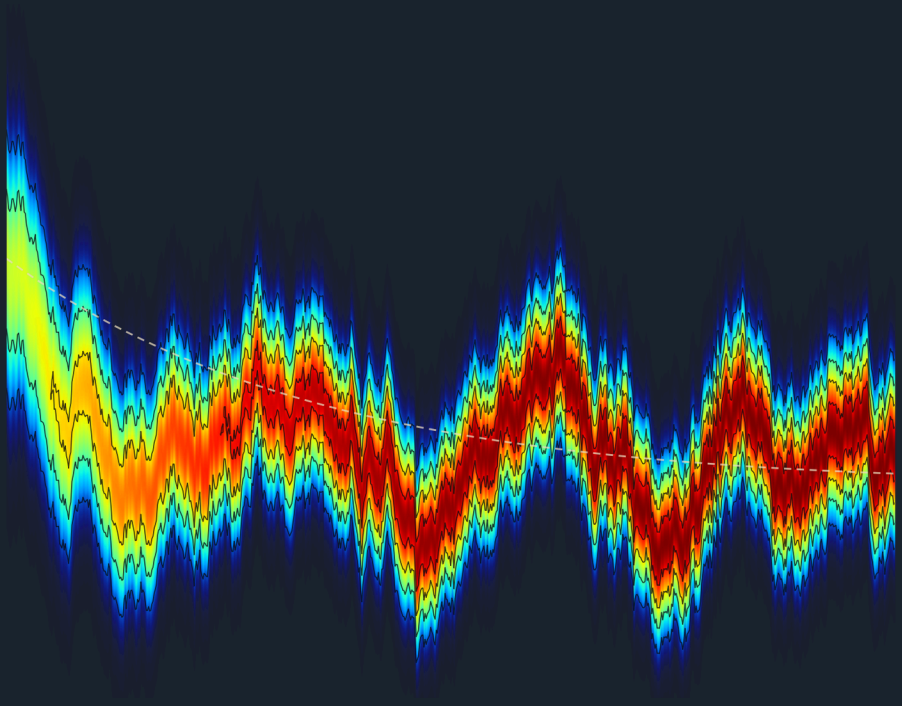


A crowd's density, carried by one path of a common noise.

Mean Field Games

Theory and Algorithms, from N-Player Games to the Master Equation

Nikolaos Vardakis

ODIN PRESS

Mean Field Games

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Preface

A crowd in equilibrium is a fixed point of two equations that run in opposite directions in time. One is a backward Hamilton–Jacobi–Bellman equation for the value of a single player, solved from the horizon; the other is a forward Fokker–Planck equation for the law of the crowd, solved from the start. Each player optimises against the crowd, and the crowd is nothing but the players: the value needs the law, the law needs the feedback that the value produces, and an equilibrium is a pair that answers both. This book is about that pair, from the moment it appears as the limit of a game of N players to the moment it is computed, and about the single equation, the master equation, in which the two are folded into one.

It is written for graduate students and for quantitative professionals who want the proofs whole and the algorithms precise. It assumes stochastic calculus (Brownian motion, martingales, Itô's formula and stochastic differential equations), the Hamilton–Jacobi–Bellman equation of stochastic control with its verification theorem, and basic partial differential equations: the heat equation, weak convergence of measures, a fixed point theorem. The control theory of one player is taken as known and stated with a citation; everything the crowd adds to it is proved.

The book runs in seven chapters. Chapter 1 derives the system from a symmetric game of N players, through the empirical measure, propagation of chaos and one player facing a frozen crowd. Chapter 2 proves that the system has a solution when the coupling smooths the crowd, solves the running model in closed form, and finds the stationary crowd and its long-horizon turnpike. Chapter 3 asks when the equilibrium is unique, proves the Lasry–Lions theorem, locates the horizons at which uniqueness fails, and prices the equilibrium against a planner who moves the whole crowd. Chapter 4 reads the same system as a McKean–Vlasov forward–backward stochastic differential equation and builds the derivative on the space of measures. Chapter 5 writes one equation for every initial law, adds a noise

common to all players, and proves that the equilibria of N players converge to the limit at the rate 1/N. Chapter 6 computes equilibria on a grid and says when each of four solvers converges. Chapter 7 carries the frame into four markets: banks that lend to one another, traders who sell together, households at a borrowing limit, and producers of a resource that runs out.

One model returns in every chapter: the linear–quadratic crowd, in which each player pays for its effort and for its distance to a target set by the crowd's mean. Everything about it can be solved by hand. From a Gaussian start its value is quadratic, its feedback affine and its law Gaussian, and its equilibrium reduces to a Riccati equation and a boundary problem for one real function. It is the model on which each general theorem is tested, and it is also the model that keeps falling outside the general theorems' hypotheses; where it does, the chapter says so plainly and proves the result from the closed forms. I have learned to trust a theorem only after its simplest case has been solved by hand, and this model is that case for the whole subject. It also shows, more than once, that the order of limits matters: a crowd large compared with everything in the problem is not the same object as a game whose horizon grows first.

The algorithms are in the chapters, as pseudocode boxes with one step per line, complete enough to be implemented as they stand, and the tables of Chapter 6 report what they did when they were run.

The thread of the book is one I have followed for many years: the place where two theories turn out to be one. Here the backward equation of a player and the forward equation of a crowd become a single fixed point, and the analytic and probabilistic routes to it describe the same object, row by row.

Nikolaos Vardakis

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Chapter 1

From N Players to a Crowd

A crowd is many players, each solving its own problem against all the others. A trader working off a large order, a bank managing its reserves, a producer drawing down a finite stock, a household saving against bad years: each chooses how hard to act, and each pays a cost that depends on what the rest of the crowd does. With a handful of players the best responses form a Nash equilibrium of a stochastic differential game, one equation per player, all coupled. With many players, none of whom matters alone, the game simplifies, and the simplification is the subject of this book. One representative player solves a control problem against the law of the crowd, and the law of the crowd must be the law that this player's optimal state produces. The first statement is a backward Hamilton–Jacobi–Bellman equation for a value u; the second is a forward Fokker–Planck equation for a law m. An equilibrium is a fixed point of the pair: two equations, one crowd.

The book follows that fixed point. This chapter derives it from a game of N players. Chapter 2 proves that it exists, Chapter 3 asks when it is unique and what it costs against a planner, Chapter 4 reads it as a forward–backward stochastic differential equation, Chapter 5 writes one equation for every initial law and proves the limit from N players, Chapter 6 computes it, and Chapter 7 carries it into four markets. One running model, the linear–quadratic crowd, returns in every chapter, because everything about it can be written down and checked by hand.

This chapter delivers the mean field game system with its derivation, the definition of a mean field equilibrium, and the linear–quadratic crowd's Gaussian equilibrium in closed form beside its explicit N-player Nash equilibrium. It runs in six steps: a game of N players; the empirical measure through which a symmetric game sees the others; propagation of chaos, which makes that random measure deterministic; one player against a frozen crowd; the consistency condition that closes the loop; and the linear–quadratic crowd, where every step is computed.

1.1 A Game with Many Players

Fix a horizon $T > 0$ and N players, and let player i steer a state $X_t^i \in \mathbb{R}^d$ through its own drift while paying for where it stands among the others. One probability setting carries the whole book.

Assumption 1.1 (The standing probability setting)**.** A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions carries independent d-dimensional $(\mathcal{F}_t)$-Brownian motions $W^1, \dots, W^N$, started at the origin, and independent $\mathcal{F}_0$-measurable initial states $X_0^1, \dots, X_0^N$ with a common law m_0 of finite second moment. The horizon $T > 0$ and the noise level $\sigma > 0$ are constants. Expectation under $\mathbb{P}$ is written $\mathbb{E}$.

The usual conditions, completeness and right-continuity of the filtration, are the frame in which every stochastic integral of the book is defined without qualification. Since each W^i is an $(\mathcal{F}_t)$-Brownian motion, its increments are independent of $\mathcal{F}_0$, and the initial states are independent of all the noise.

Definition 1.2 (N-player game)**.** Player $i \in \{1, \dots, N\}$ chooses an admissible control α^i, a progressively measurable process with values in $\mathbb{R}^d$ and $\mathbb{E}\int_0^T |\alpha_t^i|^2\, dt < \infty$, and its state solves $dX_t^i = \alpha_t^i\, dt + \sigma\, dW_t^i$ from X_0^i. With $X_t := (X_t^1, \dots, X_t^N)$, player i pays

$$J^i(\alpha^1, \dots, \alpha^N) := \mathbb{E}\Big[\int_0^T F^i(X_t, \alpha_t^i)\, dt + G^i(X_T)\Big],$$

where $F^i \colon (\mathbb{R}^d)^N \times \mathbb{R}^d \to \mathbb{R}$ and $G^i \colon (\mathbb{R}^d)^N \to \mathbb{R}$ are measurable and, for a constant C, satisfy $-C(1 + |x|^2) \leq F^i(x, a) \leq C(1 + |x|^2 + |a|^2)$ and $|G^i(x)| \leq C(1 + |x|^2)$. The N state equations together with the N costs form the N-player game.

The N-player game is well posed because each cost is finite. The state is explicit, $X_t^i = X_0^i + \int_0^t \alpha_s^i \, ds + \sigma W_t^i$, so the Cauchy–Schwarz inequality and Doob's maximal inequality $\mathbb{E}\sup_{t\le T}|W_t^i|^2 \le 4dT$ bound $\mathbb{E}\sup_{t\le T}|X_t^i|^2$ by $3\,\mathbb{E}|X_0^i|^2 + 3T\,\mathbb{E}\int_0^T |\alpha_t^i|^2\,dt + 12\sigma^2 dT$, and the two bounds on F^i make both parts of the integrand integrable. The players interact only through their costs: the drift of player i is its own control and its noise is its own. The dependence of F^i on the whole vector x is so far arbitrary, and the next definition restricts it.

Definition 1.3 (Symmetric game)**.** The N-player game is symmetric when there are functions F and G, invariant under every permutation of their last $N-1$ arguments, with $F^i(x,a) = F(x^i, a, (x^j)_{j\neq i})$ and $G^i(x) = G(x^i, (x^j)_{j\neq i})$ for every player i: each player pays the same function of its own state, its own control and the unordered collection of the others' states.

A symmetric game makes the players interchangeable, since one F and one G serve all of them, and makes each player indifferent to the names of the others. Cardaliaguet's notes [1] start the theory here, and Carmona and Delarue's monograph [2] develops the general theory of N-player stochastic differential games around it.

Example 1.4 (Symmetric and non-symmetric costs)**.** Let V and Λ be continuous functions on $\mathbb{R}^d$ with quadratic growth. The pairwise cost $F^i(x,a) = \frac{1}{2}|a|^2 + V(x^i) + \frac{1}{N-1}\sum_{j\neq i}\Lambda(x^i - x^j)$ is symmetric, since permuting the others permutes the terms of the sum. Symmetric too is the dispersion cost $F^i(x,a) = \frac{1}{2}|a|^2 + \frac{r}{2}\cdot\frac{1}{N}\sum_j |x^j - \bar{x}|^2$, with $\bar{x} := \frac{1}{N}\sum_j x^j$ and $r \ge 0$, which charges every player for the empirical variance of the whole crowd; it is invariant under all permutations of the N states. A leader breaks the symmetry. If player 1 pays $\frac{1}{2}|a|^2 + \frac{q_0}{2}|x^1 - z|^2$ to track a fixed point z and every player $i \ge 2$ pays $\frac{1}{2}|a|^2 + \frac{q}{2}|x^i - x^1|^2$ to follow it, exchanging the states of players 1 and 2 changes the cost of player 3, and no single F serves the leader and its followers.

The leader keeps one distinguished state however many followers there are; the crowds of this book are anonymous. Within a symmetric game the notion of equilibrium must still be chosen, because a deviation can leave the others' behaviour fixed in two senses.

Definition 1.5 (Nash equilibrium)**.** A Nash equilibrium is an admissible profile $\alpha^* = (\alpha^{*1}, \dots, \alpha^{*N})$ such that $J^i(\alpha^*) \le J^i(\beta, \alpha^{*,-i})$ for every player

Chapter 3

Uniqueness and Its Price

A crowd in equilibrium is a flow of the population that reproduces itself: every player answers the flow optimally, and the answers, taken together, are the flow again. Chapter 1 wrote this fixed point as two equations that run in opposite directions in time. Chapter 2 proved that it exists when the coupling is bounded and smoothing, and wrote the linear–quadratic crowd's equilibrium in closed form whenever its uniqueness function D does not vanish at the horizon. Neither chapter asked the two questions that decide whether an equilibrium can serve as a prediction. Is there only one? And is it any good, measured against what a planner who moves the whole crowd at once would achieve?

This chapter answers both. Uniqueness is a property of the coupling, not of the existence proof. We prove the Lasry–Lions theorem, which turns a dislike of crowded places into uniqueness through a duality between the two equations, and we find exactly where the herding crowd, $\theta > 1$, loses uniqueness: at isolated critical horizons T_n, where its mean resonates. Published games with two and three equilibria show what genuine multiplicity looks like. The price is measured against the planner's problem, whose optimality system is the game's own with one extra term; for the running model it is the game's system with θ replaced by $\theta(2-\theta)$, and the price of anarchy PoA comes out in closed form.

§3.1 settles the running model at every horizon and sets beside it the games with two and three equilibria. §3.2 proves the Lasry–Lions theorem. §3.3 reads equilibria as critical points of a potential and meets the critical horizon again, as an old acquaintance from the calculus of variations. §3.4 solves the planner's problem, §3.5 prices the difference, and §3.6 introduces

displacement monotonicity, which covers the flocking crowd where the first monotonicity fails.

3.1 Two Equilibria of One Game

Chapter 2 wrote the linear–quadratic crowd's equilibrium whenever $D(T) \neq 0$; this section settles every horizon, including those where $D(T) = 0$ and the count of equilibria changes. On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ each player controls $dX_t = \alpha_t\,dt + \sigma\,dW_t$ with $\sigma > 0$, starts from $X_0 \sim m_0 = \mathcal{N}(\bar{x}_0, s_0^2)$ with $s_0 > 0$, and, against a flow (m_t) with mean $\bar{m}_t := \int x\, m_t(dx)$, pays

$$J(\alpha; m) = \mathbb{E}\left[\int_0^T \left(\tfrac{1}{2}\alpha_t^2 + \tfrac{q}{2}(X_t - \theta\bar{m}_t)^2\right) dt + \tfrac{c}{2}(X_T - \theta\bar{m}_T)^2\right],$$

with $q \geq 0$, $c \geq 0$ and $\theta \in \mathbb{R}$. In the notation of Chapter 1, $H(x,p) = \frac{1}{2}p^2$, $f(x,m) = \frac{q}{2}(x-\theta\bar{m})^2$ and $g(x,m) = \frac{c}{2}(x-\theta\bar{m})^2$. Two objects of the earlier chapters carry the noise and do not see θ: the fluctuation Riccati solution η, with $-\partial_t\eta_t + \eta_t^2 = q$ and $\eta_T = c$, equal to $\eta_t = \partial_\tau E(T-t)/E(T-t)$ with $E(\tau) := \cosh(q^{1/2}\tau) + (c/q^{1/2})\sinh(q^{1/2}\tau)$ (and $E(\tau) = 1 + c\tau$ when $q = 0$); and the variance s_t^2, with $\partial_t s_t^2 = -2\eta_t s_t^2 + \sigma^2$ from s_0^2.

The first result, the reduction of §1.6 with the law of §2.4 included, removes the crowd from the problem.

Proposition 3.1 (The mean decides the equilibrium)**.** Let $q, c \geq 0$ and $\theta \in \mathbb{R}$. A flow (m_t), continuous from $[0,T]$ into $\mathcal{P}_2(\mathbb{R})$, is a mean field equilibrium of the linear–quadratic crowd if and only if $m_t = \mathcal{N}(\bar{m}_t, s_t^2)$ for every t, where the mean is a C^2 solution of the two-point problem

$$\partial_{tt}\bar{m}_t = q(1-\theta)\,\bar{m}_t \quad \text{on } [0,T],$$

with $\bar{m}_0 = \bar{x}_0$ and $\partial_t\bar{m}_T = c(\theta-1)\bar{m}_T$. The equilibrium feedback is $\hat{\alpha}(t,x) = -\eta_t(x - \bar{m}_t) + \partial_t\bar{m}_t$, and equilibria correspond one to one to solutions of the two-point problem.

Proof. Combine two earlier results. The theorem of §1.6 on the mean field equilibria of the linear–quadratic crowd, which holds for every θ and every initial law in $\mathcal{P}_2(\mathbb{R})$, puts the equilibria in one-to-one correspondence with the C^2 solutions of the two-point problem, through the feedback $\partial_t\bar{m}_t - \eta_t(x - \bar{m}_t)$. Under that feedback $d(X_t - \bar{m}_t) = -\eta_t(X_t - \bar{m}_t)\,dt + \sigma\,dW_t$,

whatever the mean path, so, as in the proof of the closed-form theorem of §2.4, $X_t - \bar{m}_t$ is a centred Gaussian variable with variance s_t^2, and $m_t = \mathcal{N}(\bar{m}_t, s_t^2)$. A flow determines its mean and conversely, so the correspondence is one to one. □

What is left is a linear boundary problem for one real function, with one condition at each end of $[0, T]$. Chapter 1's example of two directions of time showed what such problems can do: one solution, none, or infinitely many, according to the horizon. The function that decides is the uniqueness function D of §2.4, and shooting from $t = 0$ makes this exact. Let D_0 and S solve $\partial_{\tau\tau} y = q(1-\theta)y$ with $D_0(0) = 1$, $\partial_\tau D_0(0) = 0$ and $S(0) = 0$, $\partial_\tau S(0) = 1$: the cosine-type and the sine-type solutions. Then $D = D_0 + c(1-\theta)S$ is the solution with $D(0) = 1$ and $\partial_\tau D(0) = c(1-\theta)$.

Lemma 3.2 (Three solutions of one equation)**.** The functions D_0, S and D satisfy $\partial_\tau S = D_0$, $\partial_\tau D_0 = q(1-\theta)S$ and $D_0^2 - q(1-\theta)S^2 = 1$. Consequently $\partial_\tau S + c(1-\theta)S = D$ and $\partial_\tau D_0 + c(1-\theta)D_0 = \partial_\tau D$; D and $\partial_\tau D$ have no common zero; and if $\theta \le 1$, then $D \ge 1$ and $\partial_\tau D \ge 0$ on $[0, \infty)$.

Proof. Differentiate the equation: $\partial_\tau S$ solves it, with value 1 and slope $q(1-\theta)S(0) = 0$ at $\tau = 0$, the data of D_0, so $\partial_\tau S = D_0$ by uniqueness for linear equations. Likewise $\partial_\tau D_0$ solves it with value 0 and slope $q(1-\theta)$, the data of $q(1-\theta)S$. The Wronskian $D_0\,\partial_\tau S - S\,\partial_\tau D_0 = D_0^2 - q(1-\theta)S^2$ has zero derivative and equals 1 at $\tau = 0$. The two sums follow from $D = D_0 + c(1-\theta)S$. If $D(\tau_0) = \partial_\tau D(\tau_0) = 0$, then D solves a linear second-order equation with zero data at τ_0, so $D \equiv 0$, against $D(0) = 1$. Let $\theta \le 1$, so that $q(1-\theta) \ge 0$, and let $\tau^* \in (0, \infty]$ be the first zero of D. On $[0, \tau^*)$ we have $\partial_{\tau\tau} D = q(1-\theta)D \ge 0$, so $\partial_\tau D \ge \partial_\tau D(0) = c(1-\theta) \ge 0$ and $D \ge D(0) = 1$. If τ^* were finite, continuity would give $D(\tau^*) \ge 1$; hence $\tau^* = \infty$. □

For $\theta \le 1$ the lemma already settles everything: D never vanishes. For $\theta > 1$ the equation is an oscillator, and its zeros get a name.

Definition 3.3 (Critical horizons of the herding crowd)**.** Let the uniqueness function D solve $\partial_{\tau\tau} D = q(1-\theta)D$, $D(0) = 1$, $\partial_\tau D(0) = c(1-\theta)$. For $\theta > 1$ and $q > 0$, with $\omega := (q(\theta-1))^{1/2}$,

$$D(\tau) = \cos(\omega\tau) - \frac{c(\theta-1)}{\omega}\sin(\omega\tau),$$

When every player of a large crowd solves a control problem against all the others, two equations appear: a backward Hamilton–Jacobi–Bellman equation for one player's value and a forward Fokker–Planck equation for the crowd. A mean field equilibrium is the fixed point of the pair, and everything else in the theory is a statement about that coupling.

This book follows the fixed point from a game of N players to the master equation. It proves when the equilibrium exists, when it is unique and what it costs against a planner, reads it again as a forward–backward stochastic equation, writes one equation for every initial law, and proves that the N-player game converges to it. It then computes equilibria, with four solvers and the conditions under which each converges, and carries the frame into four markets: systemic risk, traders executing at once, producers of an exhaustible resource and the wealth distribution.

One running model, the linear–quadratic crowd, is solved by hand in every chapter. Every main result is proved, and every algorithm is set out step by step.

Nikolaos Vardakis is a probabilist who works on optimal stopping and stochastic control, and solves the simplest case by hand before he trusts the general one.

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